Empirical Methods
Time Series Econometrics I & II
Spectral Analysis

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Definition

A function $F : [-\pi, \pi] \to \mathbb{C}^{n\times n}$ is called positive semi-definite, symmetric distribution function if it fulfills the following properties:

(i) $F(-\pi) = 0$

(ii) $F(\lambda_2) - F(\lambda_1)$ is positive semi-definite for $\lambda_2 \geq \lambda_1$

(iii) $F(\cdot)$ is right-continuous

(iv) $F(-\lambda)' = F(\pi-) - F(\lambda-)$ for $-\pi < \lambda < \pi$ and $F(\pi) \in \mathbb{R}^{n\times n}$

Notes:

- A matrix $A \in \mathbb{C}^{n\times n}$ is said to be positive semi-definite, if it is Hermitian and if $z^* A z \geq 0$ for all $z \in \mathbb{C}^n$.
- We use the short-hand notation $F(\lambda-) := \lim_{x\uparrow\lambda} F(x)$.

- The definition implies that $F(\lambda)$ is positive semi-definite for all $\lambda \in [-\pi, \pi]$.
- The matrix $F(\lambda)$ is Hermitian, i. e., $F(\lambda) = F(\lambda)^*$.
- The diagonal elements $F_{ii}(\cdot)$ are distribution functions of positive (real-valued) measures $\mu_{F_{ii}}((a, b]) := F_{ii}(b) - F_{ii}(a)$.
- The off-diagonal elements $F_{ij}(\cdot)$, $i \neq j$ are distribution functions of complex valued measures $\mu_{F_{ij}}((a, b]) := F_{ij}(b) - F_{ij}(a)$.
- Note that a complex measure μ can be decomposed as $\mu = \mu_R + i\mu_I$ with μ_R and μ_I denoting real valued measures.

- For a univariate measurable function $a : [-\pi, \pi] \to \mathbb{C}$ the integral:

$$\int_{-\pi}^{\pi} a(\lambda)\, dF(\lambda)$$

is defined component-wise, i. e.,:

$$\int_{-\pi}^{\pi} a(\lambda)\, dF(\lambda) := \left(\int_{-\pi}^{\pi} a(\lambda)\, dF_{ij}(\lambda) \right)_{i,j} := \left(\int_{-\pi}^{\pi} a(\lambda)\, \mu_{F_{ij}}(d\lambda) \right)_{i,j}$$

with $i, j = 1, \ldots, n$.

- Consequently, the distribution function $F(\cdot)$ defines a matrix-valued measure μ_F on the Borel sets $\mathcal{B}$ of $[-\pi, \pi]$:

$$U \in \mathcal{B} \mapsto \mu_F(U) = \int_{-\pi}^{\pi} \mathbb{1}_U(\lambda)\, dF(\lambda)$$

Fourier Representation of ACFs

Positive Semi-Definite, Symmetric Distribution Function

- Since $\mu_F(U)$ is positive semi-definite for all $U \in \mathcal{B}$, it is called positive semi-definite.
- For non-zero $a \in \mathbb{C}^{1\times n}$, it holds that $F_{aa}(\lambda) := aF(\lambda)a^*$ is a distribution function of a positive measure.
- Thus, it follows that:

$$a\mu_F(\lambda)a^* = a\left(\int_{-\pi}^{\pi} \mathbb{1}_U(\lambda)\, dF(\lambda)\right) a^* = \int_{-\pi}^{\pi} \mathbb{1}_U(\lambda)\, dF_{aa}(\lambda) \geq 0$$

- The trace $F^{\text{tr}}(\lambda) := \sum_{i=1}^{n} F_{ii}(\lambda)$ is the distribution function of a positive measure μ^{tr}.
- The measures μ_{ij}, corresponding to $F_{ij}(\cdot)$, are absolutely continuous with respect to μ^{tr}.

  [A measure μ is absolutely continuous with respect to another measure ν ($\mu \ll \nu$), if $\nu(U) = 0 \Rightarrow \mu(U) = 0$].

- The Radon-Nikodým theorem states that if a measure μ is absolutely continuous with respect to a measure ν, then the measure μ has a ν-density $f(\cdot)$, i. e.:

$$\mu(U) = \int_U f(\lambda)\, \nu(d\lambda)$$

- Let, therefore, $f^{\mathrm{tr}} = (f_{ij}^{\mathrm{tr}})$ be the matrix containing the corresponding μ^{tr}-densities of μ_{ij}:
    - $f^{\mathrm{tr}}(\lambda)$ is positive semi-definite
    - $\mathrm{tr}(f^{\mathrm{tr}}(\lambda)) = 1$ almost everywhere (w.r.t. μ^{tr})

- For two functions $a, b : [-\pi, \pi] \to \mathbb{C}^{1\times n}$ define:

$$\int_{-\pi}^{\pi} (a(\lambda)\, dF(\lambda)\, b(\lambda)^*) := \int_{-\pi}^{\pi} \left(a(\lambda) f^{\mathrm{tr}}(\lambda) b(\lambda)^* \right)\, dF^{\mathrm{tr}}(\lambda),$$

  if the integral on the right-hand side exists.

- In case all integrals:

$$\int_{-\pi}^{\pi} a_i(\lambda)\overline{b_j(\lambda)}\, dF_{ij}(\lambda), \quad i, j = 1, \ldots, n$$

  exist, it holds that:

$$\int_{-\pi}^{\pi} (a(\lambda)\, dF(\lambda)\, b(\lambda)^*) = \sum_{i,j=1}^{n} \int_{-\pi}^{\pi} a_i(\lambda)\overline{b_j(\lambda)}\, dF_{ij}(\lambda)$$

Theorem

A sequence $\{\Gamma(k)\}_{k\in\mathbb{Z}}$ with $\Gamma(\cdot) \in \mathbb{R}^{n\times n}$ is the autocovariance function of a weakly stationary process if and only if there exists a positive semi-definite distribution function $F : [-\pi, \pi] \to \mathbb{C}^{n\times n}$ such that:

$$\Gamma(k) = \int_{-\pi}^{\pi} e^{i\lambda k}\, dF(\lambda) \quad \forall k \in \mathbb{Z}$$

The distribution function F is unique for a given $\Gamma(\cdot)$ and is called spectral distribution function.

- If the spectral distribution function $F(\cdot)$ of $\{y_t\}_{t\in\mathbb{Z}}$ is absolutely continuous with respect to the Lebesgue measure, then there exists a function $f : [-\pi,\pi] \to \mathbb{C}^{n\times n}$ such that:

$$F(\lambda) = \int_{-\pi}^{\lambda} f(\xi)\, d\xi \quad \forall \lambda \in [-\pi,\pi]$$

- This function $f(\cdot)$ is called spectral density function of the process $\{y_t\}_{t\in\mathbb{Z}}$; it holds that:

$$\Gamma(k) = \int_{-\pi}^{\pi} e^{i\lambda k} f(\lambda)\, d\lambda \quad \forall k \in \mathbb{Z}$$

- The spectral density function is uniquely determined almost everywhere only [w.r.t. the Lebesgue measure].

Proposition

A function $f : [-\pi, \pi] \to \mathbb{C}^{n \times n}$ is the spectral density of a stationary process if and only if:

(i) $f(\cdot)$ is integrable (component-wise) [w.r.t. the Lebesgue measure]

(ii) $f(\lambda)$ is positive semi-definite almost everywhere

(iii) $f(-\lambda) = f(\lambda)'$ almost everywhere

Fourier Representation of ACFs

Corollary

If the autocovariance function $\Gamma(\cdot)$ is absolutely summable, then it holds that:

$$f(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-i\lambda k} \Gamma(k), \quad \lambda \in [-\pi, \pi]$$

is the spectral density of the corresponding stochastic process.

- The absolute summability of $\Gamma(\cdot)$ is a sufficient but not necessary condition for the existence of a spectral density.
- If $\Gamma(\cdot)$ is absolutely summable, then the partial sums:

$$\frac{1}{2\pi} \sum_{k=-q}^{q} e^{-i\lambda k} \Gamma(k)$$

converge uniformly on $[-\pi, \pi]$ to $f(\cdot)$. This is a direct consequence of the so-called Weierstrass criterion.

Fourier Representation of ACFs

- Uniform convergence of a sequence of continuous functions implies continuity of the limiting function $f(\cdot)$.
- However, not every weakly stationary process has a spectral density.
- For example, the spectral distribution function of a (univariate) harmonic process is a step function with a finite number of jumps (not absolutely continuous).
- Thus, harmonic processes do not have a spectral density (w.r.t. to the Lebesgue measure).

Fourier Representation of ACFs

- Consider a stacked process $\{z_t\}_{t\in\mathbb{Z}} = \{(x_t', y_t')'\}_{t\in\mathbb{Z}}$ and partition the spectral density (if it exists) as follows:

$$f_z(\cdot) = \begin{pmatrix} f_x(\cdot) & f_{xy}(\cdot) \\ f_{yx}(\cdot) & f_y(\cdot) \end{pmatrix}$$

- The functions $f_x(\cdot)$ and $f_y(\cdot)$ are called auto-spectra of the processes $\{x_t\}_{t\in\mathbb{Z}}$ and $\{y_t\}_{t\in\mathbb{Z}}$.
- The functions $f_{xy}(\cdot)$ and $f_{yx}(\cdot)$ are called cross-spectra or cross-spectral densities between $\{x_t\}_{t\in\mathbb{Z}}$ and $\{y_t\}_{t\in\mathbb{Z}}$.
- More generally: For an n-dimensional process $\{y_t\}_{t\in\mathbb{Z}}$ with spectral density $f(\cdot) = (f_{ij}(\cdot))_{i,j=1,\dots,n}$, the functions $f_{ii}(\cdot)$ are the auto-spectra of the component processes $\{y_{i,t}\}_{t\in\mathbb{Z}}$.
- In that case, the functions $f_{ij}(\cdot)$, $i \neq j$ are the cross-spectra between the corresponding component processes.

- A function $a : [-\pi, \pi] \to \mathbb{C}^{1\times n}$ is called square integrable with respect to F if:

$$\int_{-\pi}^{\pi} (a(\lambda)\, dF(\lambda)\, a(\lambda)^*) := \int_{-\pi}^{\pi} a(\lambda) f^{\mathrm{tr}}(\lambda) a(\lambda)^*\, dF^{\mathrm{tr}}(\lambda) < \infty$$

  [Here F is a spectral distribution function]

- If all components of $a(\cdot)$ are square integrable with respect to $F^{\mathrm{tr}}(\cdot)$, then $a(\cdot)$ is square integrable with respect to $F(\cdot)$.

- The converse is, in general, not true.

- Every positive semi-definite matrix $M \in \mathbb{C}^{n \times n}$ has a unique Hermitian, positive semi-definite root denoted by $M^{1/2}$.
  [I. e., $M^{1/2} M^{1/2} = M$ and $M^{1/2}$ positive semi-definite].
- Thus, the above quadratic form can be expressed as follows:

$$\int_{-\pi}^{\pi} \left(a(\lambda) f^{\mathrm{tr}/2}(\lambda) \right) \left(a(\lambda) f^{\mathrm{tr}/2}(\lambda) \right)^{*} \, dF^{\mathrm{tr}}(\lambda),$$

  with $f^{\mathrm{tr}/2}(\cdot)$ denoting the "root" of $f^{\mathrm{tr}}(\cdot)$.
- Hence, the function $a(\cdot)$ is square integrable w.r.t. $F(\cdot)$ if and only if the components of $\left(a(\cdot) f^{\mathrm{tr}/2}(\cdot) \right)$ [n-dimensional] are square integrable w.r.t. $F^{\mathrm{tr}}(\cdot)$.

- Let now $a, b : [-\pi, \pi] \to \mathbb{C}^{1\times n}$ be two square integrable functions with respect to $F(\cdot)$.
- Then the integral:

$$\int_{-\pi}^{\pi} (a(\lambda)\, dF(\lambda)\, b(\lambda)^*)$$
$$= \int_{-\pi}^{\pi} \left(a(\lambda) f^{\mathrm{tr}}(\lambda) b(\lambda)^*\right)\, dF^{\mathrm{tr}}(\lambda)$$
$$= \int_{-\pi}^{\pi} \left(a(\lambda) f^{\mathrm{tr}/2}(\lambda)\right) \left(b(\lambda) f^{\mathrm{tr}/2}(\lambda)\right)^*\, dF^{\mathrm{tr}}(\lambda)$$

exists [i. e., is finite].

- All linear combinations $(\alpha a(\cdot) + \beta b(\cdot))$ with $\alpha, \beta \in \mathbb{C}$ are square integrable w.r.t. $F(\cdot)$, since:

$$\int (a+b)\, dF\, (a+b)^* = \int a\, dF\, a^* + \int b\, dF\, b^* + \int a\, dF\, b^* + \int b\, dF\, a^*.$$

[Here the function arguments are suppressed for brevity.]

- We say that two – w.r.t. $F(\cdot)$ – square integrable functions $a(\cdot), b(\cdot)$ are equivalent, if:

$$\int_{-\pi}^{\pi} \left[(a(\lambda) - b(\lambda)))\, dF(\lambda)\; (a(\lambda) - b(\lambda))^* \right] = 0$$

- We, henceforth, consider $F(\cdot)$-equivalence classes of functions.
- On these equivalence classes the mapping:

$$(a(\cdot), b(\cdot)) \mapsto \langle a, b \rangle_F := \int_{-\pi}^{\pi} a(\lambda)\, dF(\lambda)\, b(\lambda)^*$$

  is an inner product on $\mathbb{C}$ and the corresponding norm is denoted by $\|a\|_F := \langle a, a \rangle_F^{1/2}$.
- The (complex) linear vector space of square integrable functions together with this inner product is called $\mathcal{L}^2_{\mathbb{C}}([-\pi, \pi], \mathcal{B}, F)$.

Proposition

The space $\mathcal{L}^2_{\mathbb{C}}([-\pi, \pi], \mathcal{B}, F)$ is complete, i. e., it is a Hilbert space.

Definition

Let $F(\cdot)$ be a spectral distribution function of a process $\{y_t\}_{t\in\mathbb{Z}}$. The space $H^y_F := \mathcal{L}^2_{\mathbb{C}}([-\pi, \pi], \mathcal{B}, F)$ is called frequency domain of $\{y_t\}_{t\in\mathbb{Z}}$.

- It is convenient to also consider (even for real-valued processes $\{y_t\}_{t\in\mathbb{Z}}$) the time domain over the complex numbers:

$$H^y_{\mathbb{C}} = \overline{\mathrm{sp}}\,\{y_{jt}, j = 1, \ldots, n, t \in \mathbb{Z}\} \subseteq \mathcal{L}^2_{\mathbb{C}}(\Omega, \mathcal{F}, \mathbb{P})$$

- The (real) time domain $H^y = H^y_{\mathbb{R}}$ is, by construction, a subspace of $H^y_{\mathbb{C}}$.

- Denote with u_k the k-th *row* unit vector, i. e.,:

$$u_k = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{C}^{1 \times n}$$

Theorem

The mapping $\Phi : H^y_{\mathbb{C}} \to H^y_F$, defined as:

$$y_{kt} \to u_k e^{i \cdot t},$$

with $k = 1, \ldots, n$ and $t \in \mathbb{Z}$, can be uniquely extended to a bijective isometry between $H^y_{\mathbb{C}}$ and H^y_F.

[Note that the "$\cdot$" does not denote multiplication, but denotes the function $\lambda \mapsto e^{i\lambda t}$.]

The Spectral Representation of Stationary Processes

- Since Φ is (extended to) a bijective isometry, an inverse $\Phi^{-1} : H_F^y \to H_{\mathbb{C}}^y$ exists.
- With the inverse Φ^{-1}, every function $a \in H_F(y)$ is mapped (uniquely) to an element in $H_{\mathbb{C}}^y$.
- This holds, in particular, for the row functions $u_k \mathbb{1}_A(\cdot)$ with $A \in \mathcal{B}_{[-\pi,\pi]}$.
- Thus, for all $A \in \mathcal{B}$, one can define the random vector:

$$z(A) = (z_k(A))_{k=1,\dots,n}, \quad \text{with} \quad z_k(A) = \Phi^{-1}(u_k \mathbb{1}_A)$$

- This mapping $[A \mapsto z(A)]$ is σ-additive, since Φ^{-1} is linear and continuous: Let $(A_m)_{m\in\mathbb{N}}$ be a set of mutually disjoint Borel sets, then:

$$z_k\left(\bigcup_{m=1}^{\infty} A_m\right) = \Phi^{-1}\left(\sum_{m=1}^{\infty} u_k \mathbb{1}_{A_m}\right) = \sum_{m=1}^{\infty} \Phi^{-1}(u_k \mathbb{1}_{A_m}) = \sum_{m=1}^{\infty} z_k(A_m)$$

The Spectral Representation of Stationary Processes

- Thus, one can interpret $z(\cdot)$ as a random measure[1] on $[-\pi, \pi]$.
- Consider $A_1, A_2 \in \mathcal{B}$, then:

$$\begin{aligned}
\mathbb{E}\left(z_k(A_1)\overline{z_l(A_2)}\right) &= \langle \Phi^{-1}(u_k \mathbb{1}_{A_1}), \Phi^{-1}(u_l \mathbb{1}_{A_2}) \rangle \\
&= \langle u_k \mathbb{1}_{A_1}, u_l \mathbb{1}_{A_2} \rangle_F \\
&= \int_{-\pi}^{\pi} \mathbb{1}_{A_1}(\lambda) u_k \, dF(\lambda) \, u_l^* \mathbb{1}_{A_2}(\lambda) \\
&= \mu_{kl}(A_1 \cap A_2)
\end{aligned}$$

- Combining all components leads to:

$$\mathbb{E}\left(z(A_1) z(A_2)^*\right) = \mu_F(A_1 \cap A_2)$$

- The process $z(\lambda) := z([-\pi, \lambda])$ for $\lambda \in [-\pi, \pi]$ is called the spectral process of $\{y_t\}_{t \in \mathbb{Z}}$.

[1] An exact definition of random measures is given on the next slide.

- Let $(\Omega, \mathcal{F})$ and $(E, \mathcal{E})$ denote two measurable spaces. A mapping $\mu : \Omega_1 \times \mathcal{E} \to [0, \infty)$ is called a random measure if it fulfills the following properties:
    - $\omega \mapsto \mu(\omega, A)$ is a *random variable* for every $A \in \mathcal{E}$
    - $A \mapsto \mu(\omega, A)$ is a *measure* on $(E, \mathcal{E})$ for every $\omega \in \Omega$
- Example: Let $X_1, \ldots, X_n$ denote random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i. e., $X_1, \ldots, X_n$ are $(\Omega, \mathcal{F})$ – $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ measurable functions.
- Then, the mapping $\mu_n : \Omega \times \mathcal{B}_{\mathbb{R}} \to [0, \infty)$:

$$\mu_n(\omega, A) := \frac{1}{n} \sum_{i=1}^{n} \delta_{\{X_i(\omega)\}}(A)$$

is a random measure.

- On the previous slide δ_x denotes the Dirac measure in x, i. e.,:

$$\delta_x(A) := \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

- Clearly, the set $(-\infty, x] \in \mathcal{B}_{\mathbb{R}}$ for every $x \in \mathbb{R}$. Thus, the mapping:

$$\begin{aligned} F_n(x) &:= \mu_n(\omega, (-\infty, x]) \\ &= \frac{1}{n} \sum_{i=1}^{n} \delta_{\{X_i(\omega)\}}((-\infty, x]) \\ &= \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{X_i(\omega) \leq x\}} \end{aligned}$$

  is a well-defined *random variable*.

- $F_n(x)$ is called empirical distribution function.

The Spectral Representation of Stationary Processes

- The spectral process can be interpreted as a random distribution function.
- Caution: The same notation $z(\cdot)$ is used for both the random measure and the spectral process.

Theorem

The spectral process has the following properties:

(i) $z(-\pi) = 0$ a.s. [w.r.t. $\mathbb{P}$]

(ii) $\mathbb{E}\left(z(\lambda)^* z(\lambda)\right) < \infty$ for all $\lambda \in [-\pi, \pi]$

(ii) $\mathbb{E}\left(z(\lambda)\right) = 0$

(iv) $\text{l.i.m.}_{\varepsilon \downarrow 0} z(\lambda + \varepsilon) = z(\lambda)$ for $\lambda \in [-\pi, \pi)$

(v) $\mathbb{E}\left[(z(\lambda_4) - z(\lambda_3))(z(\lambda_2) - z(\lambda_1))^*\right] = 0$ for $-\pi \leq \lambda_1 < \lambda_2 \leq \lambda_3 < \lambda_4 \leq \pi$

(vi) $\mathbb{E}\left(z(\lambda) z(\lambda)^*\right) = F(\lambda)$ for $\lambda \in [-\pi, \pi]$

(vi) $\mathbb{E}\left[(z(\lambda_2) - z(\lambda_1))(z(\lambda_2) - z(\lambda_1))^*\right] = F(\lambda_2) - F(\lambda_1)$ for $-\pi \leq \lambda_1 < \lambda_2 \leq \pi$

The Spectral Representation of Stationary Processes

Definition

A stochastic process $\{z(\lambda)\}_{\lambda \in [-\pi,\pi]}$ with complex-valued random vectors $z(\lambda) : \Omega \to \mathbb{C}^n$ is called orthogonal increment process if properties (i)–(v) of the above theorem are fulfilled.

- The spectral process allows to describe Φ^{-1} explicitly.
- Start by considering simple functions $a \in H_F^y$ of the form:

$$a(\lambda) = \sum_{m=1}^{M} a_m \mathbb{1}_{A_m}(\lambda) \quad \text{with} \quad a_m = (a_{1m}, \dots, a_{nm}) \in \mathbb{C}^{1\times n}, A_m \in \mathcal{B}$$

- Since Φ^{-1} is a linear mapping, it follows that:

$$\Phi^{-1}(a(\lambda)) = \Phi^{-1}\left(\sum_{m=1}^{M}\left(\sum_{k=1}^{n} a_{km} u_k \mathbb{1}_{A_m}(\lambda)\right)\right)$$

The Spectral Representation of Stationary Processes

- Let $\mathcal{S} \subset \mathcal{L}^2([-\pi, \pi], \mathcal{B}, \mu_F)$ be the set of all measurable, simple functions. It can be shown that $\mathcal{S}$ is dense in $\mathcal{L}^2([-\pi, \pi], \mathcal{B}, \mu_F)$.
- Thus, every function $a(\cdot) \in H_F^y$ can be approximated by simple functions, i. e., there exists $(a^{(k)})_{k \in \mathbb{N}}$ with $a^{(k)} \in H_F^y$, such that $\lim_k a^{(k)}(\cdot) = a(\cdot)$.

  [Limit according to the $\|\cdot\|_F$ norm, i. e. $\|a^{(k)} - a\|_F \to 0$ as $k \to \infty$.]
- Continuity of Φ^{-1}, therefore, implies that:

$$\Phi^{-1}(a) = \text{l.i.m.}_{k \to \infty} \Phi^{-1}(a^{(k)})$$

- Based on the above construction, we call the mapping Φ^{-1} stochastic integral with respect to z and use the notation:

$$\Phi^{-1}(a) = \int_{-\pi}^{\pi} a(\lambda)\, dz(\lambda)$$

The Spectral Representation of Stationary Processes

- It is often convenient to work with random vectors containing components in $H^y_{\mathbb{C}}$.
- Correspondingly, consider matrix-valued functions $a : [-\pi, \pi] \to \mathbb{C}^{n\times n}$ with rows $a_k \in H^y_F$ for $k = 1, \dots, n$.
- Some notation and definitions:

$$y_t = \begin{pmatrix} y_{it} \\ \vdots \\ y_{nt} \end{pmatrix} \in (H^y_{\mathbb{C}})^n \quad \iff \quad y_{it} \in H^y_{\mathbb{C}} \quad i = 1, \dots, n$$

- For a matrix-valued function $a : [-\pi, \pi] \to \mathbb{C}^{m\times n}$, $a \in (H^y_F)^m$ if all rows $(a_k(\cdot))_{k=1,\dots,m}$ are elements of H^y_F.

The Spectral Representation of Stationary Processes

- For random vectors $y \in (H_{\mathbb{C}}^{y})^m$ and matrices $a \in (H_F^y)^m$ define $\Phi(y)$ and $\Phi^{-1}(a)$ component-wise and row-wise, respectively, i. e.,:

$$\Phi(y) := (\Phi(y_1)', \ldots, \Phi(y_m)')'$$

and:

$$\Phi^{-1}(a) = \int_{-\pi}^{\pi} a(\lambda)\, dz(\lambda) := \begin{pmatrix} \Phi^{-1}(a_1) \\ \vdots \\ \Phi^{-1}(a_m) \end{pmatrix} = \begin{pmatrix} \int_{-\pi}^{\pi} a_1(\lambda)\, dz(\lambda) \\ \vdots \\ \int_{-\pi}^{\pi} a_m(\lambda)\, dz(\lambda) \end{pmatrix}$$

- If $a(\lambda) = a_0(\lambda) I_n$ with $a(\cdot)$ being a scalar, complex valued function, simply write:

$$\Phi^{-1}(a) = \int_{-\pi}^{\pi} a(\lambda)\, dz(\lambda) = \int_{-\pi}^{\pi} a_0(\lambda)\, dz(\lambda)$$

The Spectral Representation of Stationary Processes

- The "inner product" of two random vectors $v, w \in (H_{\mathbb{C}}^{y})^n$ is defined as the matrix:

$$\langle v, w \rangle := \left(\langle v_k, w_l \rangle \right)_{k,l} = \mathbb{E}(vw^*)$$

- Note that the mapping $\langle \cdot, \cdot \rangle$ is not an inner product in the usual sense as it maps into $\mathbb{C}^{n \times n}$. However, every component of the resulting matrix is an inner product.

- For $a, b \in (H_F^y)^n$ set:

$$\langle a, b \rangle_F := \left(\langle a_k, b_l \rangle_F \right)_{k,l} := \int_{-\pi}^{\pi} (a(\lambda)\, dF(\lambda)\, b(\lambda)^*)$$

The Spectral Representation of Stationary Processes

- Using this notation and the isometry property of Φ^{-1}, i. e.,:

$$\langle \Phi^{-1}(a), \Phi^{-1}(b) \rangle = \langle a, b \rangle_F$$

leads to:

$$\mathbb{E}\left[\int_{-\pi}^{\pi} (a(\lambda)\, dz(\lambda))\, (b(\lambda)\, dz(\lambda))^* \right] = \int_{-\pi}^{\pi} (a(\lambda)\, dF(\lambda)\, b(\lambda)^*)$$

- If a spectral density $f(\cdot)$ exists, the above expression can be rewritten as:

$$\mathbb{E}\left[\int_{-\pi}^{\pi} (a(\lambda)\, dz(\lambda))\, (b(\lambda)\, dz(\lambda))^* \right] = \int_{-\pi}^{\pi} a(\lambda) f(\lambda) b(\lambda)^*\, d\lambda$$

The Spectral Representation of Stationary Processes

- Consider in particular $a(\lambda) = e^{i\lambda t} I_n$. Then, the integral representation of $\Phi^{-1}(a(\lambda))$:

$$y_t = \int_{-\pi}^{\pi} e^{i\lambda t}\, dz(\lambda)$$

  is the spectral representation of the process $\{y_t\}_{t\in\mathbb{Z}}$.

- The function $e^{i\lambda t}$ can be (uniformly) approximated by step functions:

$$e^{i\lambda t} \approx \sum_{j=0}^{M-1} e^{i\lambda_j^M t} \mathbb{1}_{(\lambda_j^M, \lambda_{j+1}^M]}(\lambda), \quad \lambda_j^M = -\pi + \frac{2\pi j}{M},\ j = 0, \ldots, M$$

- This leads to:

$$y_t = \int_{-\pi}^{\pi} e^{i\lambda t}\, dz(\lambda) = \text{l.i.m.}_{M\to\infty} \sum_{j=0}^{M-1} e^{i\lambda_j^M t} \left(z(\lambda_{j+1}^M) - z(\lambda_j^M) \right), \quad t \in \mathbb{Z}$$

The Spectral Representation of Stationary Processes

Theorem

For every (zero-mean) weakly stationary process $\{y_t\}_{t \in \mathbb{Z}}$, there exists an orthogonal increments process $\{z(\lambda) \mid \lambda \in [-\pi, \pi]\}$ such that:

$$
\begin{aligned}
y_t &= \int_{-\pi}^{\pi} e^{i\lambda t} dz(\lambda) \\
&= \text{l.i.m.}_{M \to \infty} \sum_{j=0}^{M-1} e^{i\lambda_j^M t} (z(\lambda_{j+1}^M) - z(\lambda_j^M))
\end{aligned}
$$

for all $t \in \mathbb{Z}$.

The so-called spectral process is almost surely [w.r.t. to the Lebesgue measure] unique and $z_j(\lambda) \in H_{\mathbb{C}}^y$ for $j = 1, \ldots, n$.

The Spectral Representation of Stationary Processes

- The spectral representation theorem states that every weakly stationary process $\{y_t\}_{t\in\mathbb{Z}}$ can be approximated [to arbitrary accuracy] via a sequence of harmonic processes.

- However, this approximation is not uniform, i. e., generally the sequence:

$$\sup_t \mathbb{E}(y_t - y_t^M)^*(y_t - y_T^M)$$

  does not converge to zero as $M \to \infty$.

- This explains the "potential contradiction" that a (linearly) regular process can be represented as the limit of singular processes.

The Spectral Representation of Stationary Processes

- The spectral representation can also be defined for processes with non-zero expectation.
- In this case, the spectral process $z(\lambda)$ has a jump at zero, i. e., $z(0) - z(0-) \neq 0$ and $\mathbb{E}(z(0) - z(0-)) = \mathbb{E}(y_0) = \mathbb{E}(y_t)$.
- Consequently, the spectral distribution function $F(\cdot)$ also has a discontinuity at zero and, consequently, no spectral density exists.
- In the non-zero expectation case, $F(\cdot)$ is, furthermore, related to the non-centered second moments and not to the autocovariance function:

$$\mathbb{E}y_s y_0' = \Gamma(s) + (\mathbb{E}y_s)(\mathbb{E}y_0)' = \Gamma(s) + (\mathbb{E}y_0)(\mathbb{E}y_0)'$$

The Spectral Representation of Stationary Processes

- Recall the unitary operator U that generates $\{y_t\}_{t\in\mathbb{Z}}$ in the time domain [component-wise, i. e., $Uy_{i,t} = y_{i,t+1}$]:

$$Uy_t = y_{t+1} = \int_{-\pi}^{\pi} e^{i\lambda(t+1)}\, dz(\lambda) = \int_{-\pi}^{\pi} e^{i\lambda} e^{i\lambda t}\, dz(\lambda).$$

- Thus, the forward shift in the time domain corresponds to multiplication with $e^{i\lambda}$ in the frequency domain:

$$\begin{array}{ccc} H_{\mathbb{C}}^{y} & \xrightarrow{U^k} & H_{\mathbb{C}}^{y} \\ \Phi \downarrow & & \uparrow \Phi^{-1} \\ H_{F}^{y} & \xrightarrow{(e^{i\cdot k})\cdot} & H_{F}^{y} \end{array}$$

- This illustrates the beauty of spectral analysis.

The Spectral Representation of Stationary Processes

- Consider a weakly stationary process $\{y_t\}_{t \in \mathbb{Z}}$ and the frequency band $\Delta = (\lambda_1, \lambda_2] \subseteq (0, \pi)$.
- The aim is to quantify the contribution of the corresponding frequencies to this process.
- To be able to consider real components, include also the "mirrored" frequency band $\nabla := [-\lambda_2, -\lambda_1)$.
- Define the two indicator functions:

$$\mathbb{1}_1(\lambda) := \mathbb{1}_{\Delta \cup \nabla}(\lambda) \quad \text{and} \quad \mathbb{1}_2(\lambda) = 1 - \mathbb{1}_1(\lambda)$$

- The process $\{y_t\}_{t \in \mathbb{Z}}$ can be decomposed as:

$$y_t = \int_{-\pi}^{\pi} e^{i\lambda t}\, dz(\lambda) = \underbrace{\int_{-\pi}^{\pi} \mathbb{1}_1(\lambda) e^{i\lambda t}\, dz(\lambda)}_{=:y_t^{(1)}} + \underbrace{\int_{-\pi}^{\pi} \mathbb{1}_2(\lambda) e^{i\lambda t}\, dz(\lambda)}_{=:y_t^{(2)}}$$

The Spectral Representation of Stationary Processes

- The first component, $\{y_t^{(1)}\}_{t\in\mathbb{Z}}$, is the contribution of the frequency band Δ to the process $\{y_t\}_{t\in\mathbb{Z}}$.

- The second component, $\{y_t^{(2)}\}_{t\in\mathbb{Z}}$, contains the contribution of the remaining frequencies.

- Both components are orthogonal to each other since the corresponding frequency bands are disjoint, i. e., $\mathbb{1}_1(\lambda)\mathbb{1}_2(\lambda) = 0$.

- It holds that:

$$\mathbb{E}\left(y_t^{(1)} y_t^{(2)'}\right) = \int_{-\pi}^{\pi} \mathbb{1}_1(\lambda) e^{i\lambda t} \overline{\mathbb{1}_2(\lambda) e^{i\lambda t}} \, dF(\lambda) = 0$$

The Spectral Representation of Stationary Processes

- Furthermore, it follows that:

$$\mathbb{E}\left(y_t^{(1)} y_t^{(1)'}\right) = \int_{-\pi}^{\pi} \left|\mathbb{1}_1(\lambda) e^{i\lambda t}\right|^2 dF(\lambda) = \int_{-\pi}^{\pi} \mathbb{1}_1(\lambda)\, dF(\lambda)$$
$$= (F(\lambda_2) - F(\lambda_1)) + (F((-\lambda_1)-) - F((-\lambda_2)-))$$
$$=: \Delta F + (\Delta F)'$$

- Analogously, it follows that:

$$\mathbb{E}\left(y_t^{(2)} y_t^{(2)'}\right) = \int_{-\pi}^{\pi} \left|\mathbb{1}_2(\lambda) e^{i\lambda t}\right|^2 dF(\lambda) = F(\pi) - \Delta F - (\Delta F)'$$

- Thus, the covariance matrix of $\{y_t\}_{t\in\mathbb{Z}}$ can be decomposed as:

$$\Gamma(0) = \int_{-\pi}^{\pi} (\mathbb{1}_1(\lambda) + \mathbb{1}_2(\lambda))\, dF(\lambda) = \mathbb{E}\left(y_t^{(1)} y_t^{(1)'}\right) + \mathbb{E}\left(y_t^{(2)} y_t^{(2)'}\right)$$

The Spectral Representation of Stationary Processes

- The ratio:

$$0 \le \frac{2\Delta F_{kk}}{\Gamma_{kk}(0)} \le 1$$

  is the share of the variance of the k-th component $\{y_{kt}\}_{t\in\mathbb{Z}}$ that is explained by frequencies in the frequency band Δ (or to be more precise in $\Delta \cup \nabla$).
- Frequency bands in which the (auto-)spectral distribution function F_{kk} exhibits strong increases [where ΔF_{kk} is large] are, in this sense, "important" for the process.
- When a spectral density exists, this increment can be approximated as:

$$\Delta F = \int_{\lambda_1}^{\lambda_2} f(\nu)\, d\nu \approx f(\lambda_1)(\lambda_2 - \lambda_1)$$

- The approximation $\Delta F \approx f(\lambda_1)|\Delta|$ becomes more accurate if (i) $|\Delta|$ is small and (ii) $f(\cdot)$ is more smooth.

The Spectral Representation of Stationary Processes

- The matrix $(\Delta F_{kl} + \overline{\Delta F_{kl}})$ is the covariance between $\{y_{kt}^{(1)}\}_{t\in\mathbb{Z}}$ and $\{y_{lt}^{(1)}\}_{t\in\mathbb{Z}}$ in the frequency band Δ (in fact $\Delta \cup \nabla$).
- For small intervals Δ, it holds that $\Delta F_{kl} \approx f_{kl}(\lambda_1)(\lambda_2 - \lambda_1)$ and the cross-spectral density $f_{kl}(\lambda)$ can be interpreted as a measure for the linear dependence between the components $\{y_{kt}\}_{t\in\mathbb{Z}}$ and $\{y_{lt}\}_{t\in\mathbb{Z}}$ in a "neighborhood" of the frequency λ.
- The frequencies $\lambda = 0$ and $\lambda = \pi$ have to be discussed separately. However, the previous discussion can easily be carried over to these frequencies.

- The spectral process of a univariate, real-valued, stationary, harmonic process $\{y_t\}_{t\in\mathbb{Z}}$:

$$y_t = \sum_{m=-M+1}^{M} z_m e^{i\lambda_m t}, \quad t \in \mathbb{Z},$$

with $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_M = \pi$ and $\lambda_{-m} = -\lambda_m$ is given by:

$$z(\lambda) = \sum_{m=-M+1}^{M} z_m \mathbb{1}_{\{\lambda_m \leq \lambda\}}$$

and the spectral distribution function is given by:

$$F(\lambda) = \mathbb{E}\left(|z(\lambda)|^2\right) = \sum_{m=-M+1}^{M} \mathbb{E}\left(|z_m|^2\right) \mathbb{1}_{\{\lambda_m \leq \lambda\}}$$

The Spectral Representation of Stationary Processes

Example: White Noise

- Let $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ be white noise with covariance matrix Σ. The spectral density of $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is given by:

$$f(\lambda) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\Gamma(k)e^{-i\lambda k} = \frac{1}{2\pi}\Gamma(0) = \frac{1}{2\pi}\Sigma$$

- Since the spectral density is constant, all frequencies are "equally important" for the process.
- On the contrary, let $f(\lambda) \equiv f_0$ be the (constant) spectral density of a stationary process $\{y_t\}_{t\in\mathbb{Z}}$, then:

$$\Gamma(k) = \int_{-\pi}^{\pi} f(\lambda)e^{i\lambda k}\,d\lambda = f_0\int_{-\pi}^{\pi} e^{i\lambda k}\,d\lambda = \begin{cases} 2\pi f_0 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

- Thus, a weakly stationary process is white noise if and only if it has a constant spectral density.

The Spectral Representation of Stationary Processes

- It will be seen later that every linearly regular process, i. e.,:

$$y_t = \sum_{j=0}^{\infty} B_j \varepsilon_{t-j}, \quad t \in \mathbb{Z}$$

  [cf. Wold decomposition] has a spectral density.

- Moreover, the spectral density has the following form:

$$f(\lambda) = \frac{1}{2\pi} b(\lambda) \Sigma b(\lambda)^*$$

  where $b(z) = \sum_{j=0}^{\infty} B_j z^j$.

- The converse is not true in general.

Theorem

A multivariate stationary process $\{y_t\}_{t\in\mathbb{Z}}$ with spectral density $f(\cdot)$ that has full rank almost everywhere (w.r.t. the Lebesgue measure) is regular if and only if:

$$\int_{-\pi}^{\pi} \ln(\det(f(\lambda)))\, d\lambda > -\infty$$

In this case, it holds that:

$$\det(\Sigma) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln(\det(2\pi f(\lambda)))\, d\lambda\right)$$

with Σ denoting the covariance matrix of the innovations of $\{y_t\}_{t\in\mathbb{Z}}$.

Transformations of Stationary Processes in the Time and Frequency Domains

- We consider linear, time-invariant and – in general – dynamic transformations of stationary processes.
- These transformations are also called filters or systems, the original process is called input and the transformed process output.
- Filters are methods for the extraction of certain components like noise or seasonal components [cf. the discussion of extracting trend or seasonal components in part 3].
- Linear transformations are used to build up more complicated processes (or models) from basic processes (e. g., AR, MA, ARMA from white noise).

Transformations of Stationary Processes in the Time and Frequency Domains

- For a stationary n-dimensional input process $\{x_t\}_{t\in\mathbb{Z}}$, the output process $\{y_t\}_{t\in\mathbb{Z}}$ (given finite second moments) of the form:

$$y_t := \text{l.i.m.}_{q\to\infty} \sum_{j=-q}^{q} A_j^{(q)} x_{t-j}, \quad A_j^{(q)} \in \mathbb{R}^{m\times n},\ t \in \mathbb{Z}$$

  is called linear transformation.

- The linear transformation is characterized by the so-called weighting sequence:

$$\{A_j^{(q)}, j = -q, \dots, q, q \in \mathbb{N}\}$$

- We, subsequently, assume $\{y_t\}_{t\in\mathbb{Z}}$ to be well-defined, i. e., weakly stationary and will get back to this question.

Transformations of Stationary Processes in the Time and Frequency Domains

- For the limit on the right-hand side to exist, the coefficient matrices $A_j^{(q)}$ have to fulfill assumptions (that in general depend upon $\{x_t\}_{t\in\mathbb{Z}}$).
- If the limit exists for one t, then it follows from stationarity of $\{x_t\}_{t\in\mathbb{Z}}$ that the limit exists for all $t \in \mathbb{Z}$.
- Clearly, the transformation is linear and time-invariant, i. e., $\{x_{t-s}\}_{t\in\mathbb{Z}} \to \{y_{t-s}\}_{t\in\mathbb{Z}}$ for all $s \in \mathbb{Z}$, since the cofficients do not depend on t.
- If $A_j^{(q)} = 0$ for all $j \neq 0$, $q \in \mathbb{N}$, the filter is called static, otherwise it is called dynamic.
- If $A_j^{(q)} = 0$ for all $j < 0$, $q \in \mathbb{N}$, the filter is called causal.

Transformations of Stationary Processes in the Time and Frequency Domains

Theorem

If the process $\{y_t\}_{t\in\mathbb{Z}}$ is the output of a filter [as defined on Slide 47] applied to a stationary process $\{x_t\}_{t\in\mathbb{Z}}$, then the stacked process $\{z_t\}_{t\in\mathbb{Z}} = \{({x_t}', {y_t}')'\}_{t\in\mathbb{Z}}$ is stationary and it holds that:

$$\mathbb{E}y_t = \left(\lim_{q\to\infty}\sum_{j=-q}^{q} A_j^{(q)}\right)\mathbb{E}x_t$$

$$\Gamma_{yx}(k) = \mathrm{Cov}(y_k, x_0) = \lim_{q\to\infty}\sum_{j=-q}^{q} A_j^{(q)}\Gamma_x(k-j)$$

$$\Gamma_y(k) = \mathrm{Cov}(y_k, y_0) = \lim_{q\to\infty}\sum_{j,l=-q}^{q} A_j^{(q)}\Gamma_x(k+l-j)(A_l^{(q)})'$$

Transformations of Stationary Processes in the Time and Frequency Domains

- Consider now the frequency domain H_F^y; isomorphic to the (complex) time domain $H_{\mathbb{C}}^y$.
- The weighting sequence $\{A_j^{(q)}, j = -q, \ldots, q, q \in \mathbb{N}\}$ in time domain analysis corresponds to the transfer function:

$$A(\lambda) := \lim_{q \to \infty} \sum_{j=-q}^{q} A_j^{(q)} e^{-i\lambda j}$$

in frequency domain analysis.

- The limit is defined (row-wise) with respect to the H_F^x-norm, i. e.,:

$$\lim_{q \to \infty} \int_{-\pi}^{\pi} \left(\sum_{j=-q}^{q} A_{j,l}^{(q)} e^{-i\lambda j} - A(\lambda)_l \right) dF(\lambda) \left(\sum_{j=-q}^{q} A_{j,l}^{(q)} e^{-i\lambda j} - A(\lambda)_l \right)^* = 0$$

- Here $A(\lambda)_l$ and $A_{j,l}^{(q)}$ denote the l-th row of $A(\lambda)$ and $A_j^{(q)}$, respectively.

Transformations of Stationary Processes in the Time and Frequency Domains

- It can be shown that the transfer function $A(\cdot)$ is the image of $y_0 \in (H^x_{\mathbb{C}})^n$ under the isometry Φ, i. e., $A = \Phi(y_0)$.
- It holds that:

$$y_t = \Phi^{-1}(e^{i \cdot t} A) = \int_{-\pi}^{\pi} e^{i\lambda t} A(\lambda)\, dz(\lambda)$$

- From $H^y_{\mathbb{C}} \subseteq H^x_{\mathbb{C}}$ it follows that $H^y_F \subseteq H^x_F$.
- The above equation implies that the output of the filter is uniquely determined by the transfer function.

Transformations of Stationary Processes in the Time and Frequency Domains

- Let us assume for simplicity that the process $\{x_t\}_{t\in\mathbb{Z}}$ has a spectral density $f_x(\cdot)$.
- Due to the isometry between time and frequency domain it holds that:

$$\mathbb{E}\begin{pmatrix} x_k x_0{}' & x_k y_0{}' \\ y_k x_0{}' & y_k y_0{}' \end{pmatrix} = \int_{\pi}^{\pi} e^{i\lambda k} \underbrace{\begin{pmatrix} I_n \\ A(\lambda) \end{pmatrix} f_x(\lambda) \begin{pmatrix} I_n \\ A(\lambda) \end{pmatrix}^*}_{=: f_z(\lambda)} d\lambda$$

  for all $k \in \mathbb{Z}$.
- Since the above relation holds for all $k \in \mathbb{Z}$, the spectral density $f_z(\cdot)$ of the stacked process $\{z_t\}_{t\in\mathbb{Z}} = \{(x_t{}', y_t{}')'\}_{t\in\mathbb{Z}}$ is given by:

$$f_z(\cdot) := \begin{pmatrix} f_x(\cdot) & f_{xy}(\cdot) \\ f_{yx}(\cdot) & f_y(\cdot) \end{pmatrix}$$

Theorem

If $\{x_t\}_{t\in\mathbb{Z}}$ has a spectral density $f_x(\cdot)$ and $\{y_t\}_{t\in\mathbb{Z}}$ is the output of the linear transformation:

$$y_t := \text{l.i.m.}_{q\to\infty} \sum_{j=-q}^{q} A_j^{(q)} x_{t-j}, \quad A_j^{(q)} \in \mathbb{R}^{m\times n},\ t\in\mathbb{Z}$$

then the spectral density of:

$$\{z_t\}_{t\in\mathbb{Z}} = \{({x_t}', {y_t}')'\}_{t\in\mathbb{Z}}$$

exists and and it holds that:

$$f_{yx}(\lambda) = A(\lambda) f_x(\lambda)$$
$$f_y(\lambda) = A(\lambda) f_x(\lambda) A(\lambda)^*$$

Transformations of Stationary Processes in the Time and Frequency Domains

- The above theorem implies that every stationary $\mathrm{MA}(\infty)$ process has a spectral density.

- Note that if spectral densities do not exist, then:

$$f_{xy}^{\mathrm{tr}}(\lambda) = A(\lambda) f_x^{\mathrm{tr}}(\lambda)$$
$$f_y^{\mathrm{tr}}(\lambda) = A(\lambda) f_x^{\mathrm{tr}}(\lambda) A(\lambda)^*$$

where:

$$f_z^{\mathrm{tr}}(\cdot) = \begin{pmatrix} f_x^{\mathrm{tr}}(\cdot) & f_{xy}^{\mathrm{tr}}(\cdot) \\ f_{yx}^{\mathrm{tr}}(\cdot) & f_y^{\mathrm{tr}}(\cdot) \end{pmatrix}$$

is the Radon-Nikodým derivative of the spectral distribution function of $\{z_t\}_{t\in\mathbb{Z}}$ w.r.t. to μ^{tr}, the measure corresponding to the trace of the spectral distribution function $F^{\mathrm{tr}}(\cdot)$.

- For some given weighting sequence the linear transformation may not be well-defined for all stationary processes.
- Therefore, we consider so-called l_1-filters that lead to stationary output processes for all stationary input processes.

Definition

If the weighting sequence of the linear transformation fulfills:

$$\{A_j^{(q)} = A_j, j = -q, \ldots, q, q \in \mathbb{N}\} \quad \text{with} \quad \sum_{j=-\infty}^{\infty} \|A_j\| < \infty$$

then the transformation is called l_1-filter.

Theorem

For every l_1-filter and all stationary input processes $\{x_t\}_{t\in\mathbb{Z}}$, the process $\{y_t\}_{t\in\mathbb{Z}}$ defined as:

$$y_t := \text{l.i.m.}_{q\to\infty} \sum_{j=-q}^{q} A_j x_{t-j} = \sum_{j=-\infty}^{\infty} A_j x_{t-j}, \quad t \in \mathbb{Z}$$

is stationary.

- The weighting sequence $\{A_j, j \in \mathbb{Z}\}$ of the l_1-filters is also called impulse response function or sequence.

- A filter with weighting sequence $\{A_j, j \in \mathbb{Z}\}$ can be written formally as a Laurent series:

$$A(L) = \sum_{j=-\infty}^{\infty} A_j L^j$$

with L being the lag operator, i. e., $L\{x_t\}_{t\in\mathbb{Z}} = \{x_{t-1}\}_{t\in\mathbb{Z}}$.

- Therefore:

$$\{y_t\}_{t\in\mathbb{Z}} = A(L)\{x_t\}_{t\in\mathbb{Z}} = \left(\sum_{j=-\infty}^{\infty} A_j L^j\right)\{x_t\}_{t\in\mathbb{Z}}$$

- Recall that the previously defined set $\overline{T}$ is the set of all l_1-filters.

- The transfer function of an l_1-filter $A(L) = \sum_{j=-\infty}^{\infty} A_j L^j$ is given by:

$$A(\lambda) = \sum_{j=-\infty}^{\infty} A_j e^{-i\lambda j}$$

- The above series exists as a limit in the frequency domain, but also point-wise for every $\lambda \in [-\pi, \pi]$.
- Furthermore, this convergence is uniform on $[-\pi, \pi]$, implying continuity of $A(\cdot)$. This is, again, a direct consequence of the Weierstrass criterion.
- There is a one-to-one correspondence between the weight sequence and the transfer function:

$$A_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(\lambda) e^{i\lambda j} \, d\lambda$$

- The transfer function $A(\cdot)$ is bounded, and thus:

$$\int_{-\pi}^{\pi} A(\lambda) f^{\mathrm{tr}}(\lambda) A^*(\lambda)\, dF^{\mathrm{tr}}(\lambda)$$

  exists for any stationary input process $\{x_t\}_{t\in\mathbb{Z}}$ with spectral distribution function $F(\cdot)$.

- This is the frequency domain analogue of the fact that l_1-filters generate stationary output processes from stationary inputs.

Transformations of Stationary Processes in the Time and Frequency Domains

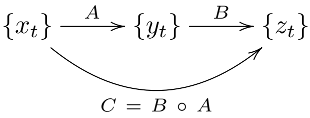
Composition of l_1-Filters

- Consider two l_1-filters $A(L)$ and $B(L)$ given by:

$$A(L) = \sum_{j=-\infty}^{\infty} A_j L^j, \quad A_j \in \mathbb{R}^{m \times n}$$

$$B(L) = \sum_{j=-\infty}^{\infty} B_j L^j, \quad B_j \in \mathbb{R}^{l \times m}$$

- The iterative application of both filters on a stationary input process leads to a stationary output process:



Theorem

Let $\{x_t\}_{t\in\mathbb{Z}}$ be a stationary process and let $\{y_t\}_{t\in\mathbb{Z}} := A(L)\{x_t\}_{t\in\mathbb{Z}}$, $\{z_t\} := B(L)\{y_t\}_{t\in\mathbb{Z}}$, with $A(L)$, $B(L)$ as given on the previous slide.

Then it holds that $\{z_t\}_{t\in\mathbb{Z}} = C(L)\{x_t\}_{t\in\mathbb{Z}}$, with weighting sequence $\{C_j\}_{j\in\mathbb{Z}}$, is given by:

$$C_j = \sum_{k=-\infty}^{\infty} B_k A_{j-k}, \quad \text{with} \sum_{j=-\infty}^{\infty} \|C_j\| < \infty$$

The corresponding transfer function $C(\lambda)$ is given by:

$$C(\lambda) = \sum_{j=-\infty}^{\infty} C_j e^{-i\lambda j} = \sum_{j=-\infty}^{\infty} B_j e^{-i\lambda j} \sum_{j=-\infty}^{\infty} A_j e^{-i\lambda j} = B(\lambda)A(\lambda)$$

- Consider a quadratic l_1-filter:

$$A(L) = \sum_{j=-\infty}^{\infty} A_j L^j, \quad A_j \in \mathbb{R}^{n \times n}$$

- An l_1-filter $B(L)$ is called the inverse filter of $A(L)$ if:

$$A(L)B(L) = B(L)A(L) = I_n L^0$$

- In this case, it holds for the transfer function that:

$$B(\lambda)A(\lambda) = A(\lambda)B(\lambda) = I_n, \quad \forall \lambda \in [-\pi, \pi]$$

- Therefore, the transfer function $A(\lambda)$ needs to be regular for all $\lambda \in [-\pi, \pi]$, i. e., $\det(A(\lambda)) \neq 0$ and $B(\lambda) = A(\lambda)^{-1}$.
- Note that $\det(A(\lambda)) \neq 0$ for all $\lambda \in [-\pi, \pi]$ implies the existence of the inverse l_1-filter.

- Consider an l_1-filter with a single (i. e., univariate) input and a single output (SISO filter).
- Rewrite the transfer function:

$$k(\lambda) = \sum_{j=-\infty}^{\infty} k_j e^{-i\lambda j}$$

  of an l_1-filter with polar coordinates as $k(\lambda) = r(\lambda)e^{i\phi(\lambda)}$.

- The function $r\colon \lambda \mapsto r(\lambda) = |k(\lambda)|$ is called gain of the filter and $\phi\colon \lambda \mapsto \phi(\lambda) = \operatorname{Im}(\ln k(\lambda))$ is called phase shift.
- Since the coefficients of the filter are real-valued, it holds that $r(\lambda) = r(-\lambda)$ and $\phi(-\lambda) = -\phi(\lambda)$.

- Consider for simplicity a univariate process $\{x_t\}_{t\in\mathbb{Z}}$ with spectral density $f_x(\lambda)$.
- As before, consider a frequency band $\Delta = (\lambda_1, \lambda_2]$ or, in fact, $\Delta \cup \nabla$.
- Recall from before that $f_x(\lambda_1)|\Delta|$, with $|\Delta| = \lambda_2 - \lambda_1$, approximates:

$$\Delta F + \Delta F' \approx 2 f_x(\lambda_1)|\Delta|,$$

  i. e., the contribution of harmonic components with frequencies in Δ (or $\Delta \cup \nabla$) to $\{x_t\}_{t\in\mathbb{Z}}$.

- Now consider $\{y_t\}_{t\in\mathbb{Z}} = A(L)\{x_t\}_{t\in\mathbb{Z}}$ for an l_1-filter $A(L)$ with gain $r(\lambda)$.
- The spectral density of $\{y_t\}_{t\in\mathbb{Z}}$ is given by:

$$f_y(\lambda) = A(\lambda)f_x(\lambda)\overline{A(\lambda)} = |A(\lambda)|^2 f_x(\lambda) = r(\lambda)^2 f_x(\lambda)$$

- Consequently:

$$f_y(\lambda_1)|\Delta| \approx r(\lambda_1)^2 f_x(\lambda_1)|\Delta|$$

- Therefore, it follows that:
  - $r(\lambda) < 1$: harmonic components in the frequency band Δ are dampened
  - $r(\lambda) > 1$: harmonic components in the frequency band Δ are amplified

- Consider the following problem: Let $\{(x_t', y_t')'\}_{t\in\mathbb{Z}}$ be a stationary process.
- Now consider the best linear approximation of $\{y_t\}_{t\in\mathbb{Z}}$ by $\{x_t\}_{t\in\mathbb{Z}}$, i. e., the solution of the minimization problem:

$$\min_{\tilde{y}_t\in(H^x)^n} \mathbb{E}\left((y_t-\tilde{y}_t)^*(y_t-\tilde{y}_t)\right)$$

  for $t\in\mathbb{Z}$.
- The Projection Theorem implies that the best linear approximation $\hat{y}_t$ is given by:

$$\hat{y}_t = P_{(H^x)^n} y_t = \text{l.i.m.}_{q\to\infty} \sum_{j=-q}^{q} K_j^{(q)} x_{t-j}$$

- Thus, one can decompose y_t as:

$$y_t = \hat{y}_t + \underbrace{y_t - \hat{y}_t}_{=:u_t} = \hat{y}_t + u_t$$

- From the projection theorem it is known that the approximation error u_t is orthogonal to $(H^x)^n$ and thus $\mathbb{E}(u_t x_s') = 0$ for all $s, t \in \mathbb{Z}$.
- Assuming that the process $\{(x_t', y_t')'\}_{t\in\mathbb{Z}}$ possesses a spectral density $f(\cdot)$, with block component $f_x(\lambda)$ invertible for (almost all) $\lambda \in [-\pi, \pi]$, implies the following result.

Theorem

Let $\{(x_t', y_t')'\}_{t\in\mathbb{Z}}$ be a stationary process with spectral density:

$$\begin{pmatrix} f_x(\lambda) & f_{xy}(\lambda) \\ f_{yx}(\lambda) & f_y(\lambda) \end{pmatrix}$$

If $f_x(\lambda)$ is positive definite for almost all $\lambda \in [-\pi, \pi]$, then the transfer function $K(\lambda)$ of the best linear approximation of $\{y_t\}_{t\in\mathbb{Z}}$ by $\{x_t\}_{t\in\mathbb{Z}}$ is given by:

$$K(\lambda) = f_{yx}(\lambda) f_x(\lambda)^{-1}$$

- The orthogonality condition $\mathbb{E}(u_{t+h}x'_t) = 0$ leads to:

$$0 = \mathbb{E}\left[\left(y_{t+h} - \text{l.i.m.}_{q\to\infty}\sum_{j=-q}^{q} K_j^{(q)} x_{t+h-j}\right) x'_t\right]$$

$$= \Gamma_{yx}(h) - \text{l.i.m.}_{q\to\infty}\sum_{j=-q}^{q} K_j^{(q)}\Gamma_x(h-j)$$

- Assuming the Wiener filter to be an l_1-filter implies:

$$\Gamma_{yx}(h) = \sum_{j=-\infty}^{\infty} K_j \Gamma_x(h-j)$$

- These equations are called Wiener-Hopf equations.

- The Wiener-Hopf equations allow calculating the coefficients K_j for given covariances.
- Let now $\{(x_t, y_t)'\}_{t\in\mathbb{Z}}$ be a bivariate process and assume that the Wiener filter is of the form:

$$K(L) = \sum_{j=0}^{N} K_j L^j$$

- In this case, the Wiener-Hopf equations reduce to a finite number of linear equations:

$$\begin{bmatrix} \Gamma_x(0) & \Gamma_x(-1) & \dots & \Gamma_x(-N) \\ \Gamma_x(1) & \Gamma_x(0) & \dots & \Gamma_x(-1-N) \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_x(N) & \Gamma_x(N-1) & \dots & \Gamma_x(0) \end{bmatrix} \begin{bmatrix} K_0 \\ K_1 \\ \vdots \\ K_N \end{bmatrix} = \begin{bmatrix} \Gamma_{yx}(0) \\ \Gamma_{yx}(1) \\ \vdots \\ \Gamma_{yx}(N) \end{bmatrix}$$

- Alternatively, the coefficients K_j can be obtained using the inverse Fourier transform:

$$K_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} K(\lambda) e^{i\lambda j} \, d\lambda$$

- A sufficient condition for the existence of this integral is that $K(\lambda) \in L^2([-\pi, \pi], \mathcal{B}, \mu)$, i. e., that every component $K_{ij}(\lambda)$ is square integrable [w.r.t. the Lebesgue measure].

- A sufficient condition for square integrability is given by:

$$\int_{-\pi}^{\pi} \operatorname{tr} \left(K(\lambda) K(\lambda)^* \right) \, d\lambda < \infty$$

- Let $\{(x_t, y_t)'\}_{t\in\mathbb{Z}}$ be a bivariate process with spectral density $f(\cdot)$ and assume that the Wiener filter is an l_1-filter.
- From the polar coordinate representation of the cross-spectrum:

$$f_{yx}(\lambda) = |f_{yx}(\lambda)|e^{i\phi(\lambda)}$$

it follows for $K(\lambda)$ that:

$$K(\lambda) = \frac{f_{yx}(\lambda)}{f_x(\lambda)} = \frac{|f_{yx}(\lambda)|}{f_x(\lambda)}e^{i\phi(\lambda)}$$

- Therefore, the gain of the Wiener filter is given by:

$$r(\lambda) = \frac{|f_{yx}(\lambda)|}{f_x(\lambda)}$$

- The coherence between two univariate processes $\{x_t\}_{t\in\mathbb{Z}}$ and $\{y_t\}_{t\in\mathbb{Z}}$ is defined as:

$$C(\lambda) := \frac{|f_{yx}(\lambda)|^2}{f_x(\lambda)f_y(\lambda)}$$

- Consider the following decomposition of the auto-spectrum $f_y(\lambda)$:

$$f_y(\lambda) = f_{\hat{y}+u}(\lambda) = f_{\hat{y}}(\lambda) + f_u(\lambda) = K(\lambda)f_x(\lambda)K(\lambda)^* + f_u(\lambda)$$

- Using this decomposition implies that:

$$C(\lambda) = \frac{f_{\hat{y}}(\lambda)}{f_{\hat{y}}(\lambda) + f_u(\lambda)} \in [0, 1]$$

- $C(\lambda)$ is the frequency domain analogue of the coefficient of determination in the linear regression model.