EC2: Macroeconomics
VC 608.145

Martin Wagner

Department of Economics

UNIVERSITÄT KLAGENFURT

Summer Term 2025

This version: July 17, 2025

This is work in progress – please check regularly for updates!

Chapter 5

Real-Business-Cycle Theory

Setting the Stage I

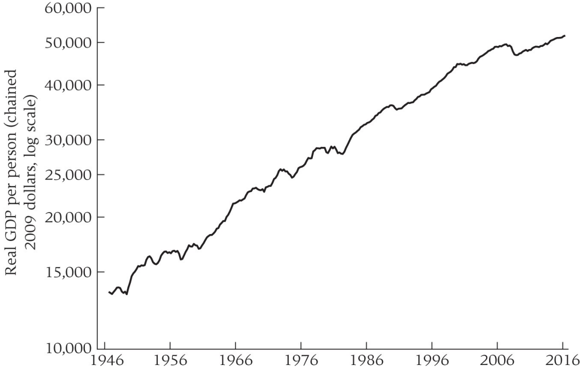

Figure 1: U.S. real GDP per person, 1947:1–2016:4; seasonally adjusted

- As discussed at the beginning: Economic activity exhibits fluctuations (around a trend).
- The fluctuations do not appear to be regular: The economy appears to be perturbed by disturbances of various types.
- Different schools of thoughts differ in their hypotheses concerning these shocks and their propagation mechanisms.

Setting the Stage II

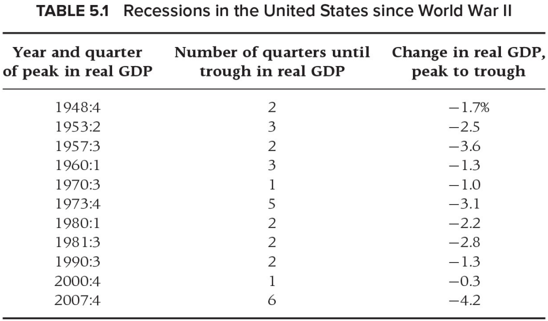

TABLE 5.1 Recessions in the United States since World War II

Year and quarter of peak in real GDP	Number of quarters until trough in real GDP	Change in real GDP, peak to trough
1948:4	2	−1.7%
1953:2	3	−2.5
1957:3	2	−3.6
1960:1	3	−1.3
1970:3	1	−1.0
1973:4	5	−3.1
1980:1	2	−2.2
1981:3	2	−2.8
1990:3	2	−1.3
2000:4	1	−0.3
2007:4	6	−4.2

- The behavior of real GDP during the eleven recessions from 1947 to 2016.
- Recessions differ in depth: –0.3% in 2000–2001 to –4.2% in the Great Recession.
- The occurrence of recessions has varied over time:
    - About one year between two recessions in the early 1980s
    - About ten years in 1960–1970 and 1991–2000

Setting the Stage III

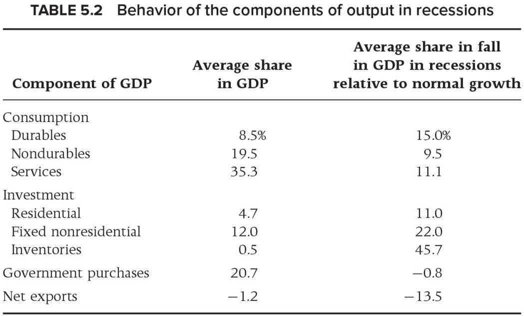

TABLE 5.2 Behavior of the components of output in recessions

Component of GDP	**Average share in GDP**	**Average share in fall in GDP in recessions relative to normal growth**
Consumption		
Durables	8.5%	15.0%
Nondurables	19.5	9.5
Services	35.3	11.1
Investment		
Residential	4.7	11.0
Fixed nonresidential	12.0	22.0
Inventories	0.5	45.7
Government purchases	20.7	−0.8
Net exports	−1.2	−13.5

- Fluctuations are distributed very unevenly across output components.
- Inventory investment (a tiny fraction) accounts for about half of the growth shortfall during recessions.
- Consumer durable goods and fixed nonresidential (business) investment feature prominently.
- Government purchases, nondurables consumption and services consumption are (relatively) stable.

Setting the Stage IV

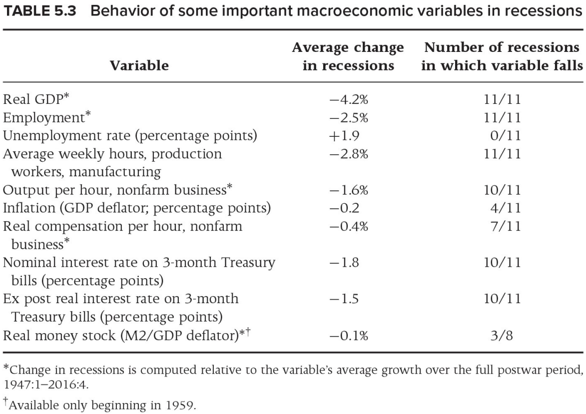

TABLE 5.3 Behavior of some important macroeconomic variables in recessions

Variable	Average change in recessions	Number of recessions in which variable falls
Real GDP*	−4.2%	11/11
Employment*	−2.5%	11/11
Unemployment rate (percentage points)	+1.9	0/11
Average weekly hours, production workers, manufacturing	−2.8%	11/11
Output per hour, nonfarm business*	−1.6%	10/11
Inflation (GDP deflator; percentage points)	−0.2	4/11
Real compensation per hour, nonfarm business*	−0.4%	7/11
Nominal interest rate on 3-month Treasury bills (percentage points)	−1.8	10/11
Ex post real interest rate on 3-month Treasury bills (percentage points)	−1.5	10/11
Real money stock (M2/GDP deflator)*†	−0.1%	3/8

*Change in recessions is computed relative to the variable's average growth over the full postwar period, 1947:1–2016:4.

†Available only beginning in 1959.

- Output growth is quite symmetric around its mean, growth is about average for relatively long and below average for brief periods only.
- During recessions employment falls and unemployment increases: Okun's law.
- Since, however, employment fluctuations are relatively small in recessions: Productivity usually falls during recessions.
- Many other variables exhibit no clear patterns over recessions.

A Solow Model with Stochastic Technology (Productivity) I

- We begin by adding stochasticity (uncertainty) to a simple model: random productivity in the Solow model. Since we focus on business cycle fluctuations in this chapter, we ignore trend growth in productivity.

- For (computational) simplicity, we again consider the Cobb-Douglas version (and denote now per-capita quantities by lower case letters):

$$y_t = \frac{Y_t}{L_t} = \frac{K_t^{\alpha}(A_t L_t)^{1-\alpha}}{L_t} = A_t^{1-\alpha} k_t^{\alpha}$$
$$k_{t+1} = \frac{1}{1+n}[(1-\delta)k_t + sA_t^{1-\alpha}k_t^{\alpha}]$$

- We assume that $A_t = \tilde{A}e^{\varepsilon_t}$, with $\tilde{A} > 0$ and ε_t i.i.d. with expectation zero and variance σ_A^2.

A Solow Model with Stochastic Technology (Productivity) II

- This leads to:

$$k_{t+1} = \frac{1}{1+n}[(1-\delta)k_t + s\tilde{A}^{1-\alpha}e^{(1-\alpha)\varepsilon_t}k_t^{\alpha}]$$

- The above equation is a nonlinear stochastic difference equation, in the case that $\sigma_A^2 = 0$, the equation "simplifies" to a (well-known from Chapter 2) deterministic equation, with a (non-stochastic) steady state:

$$k^* = \left(\frac{s\tilde{A}^{1-\alpha}}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

- It is customary to consider **approximate log-linearized** solutions of the above equation (around the non-stochastic steady state), where we consider only first-order approximations here.

A Solow Model with Stochastic Technology (Productivity) III

A Matter of Convenience

- Whilst (considering here the Cobb-Douglas case) the formulation:

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha} = \tilde{A}^{1-\alpha} e^{(1-\alpha)\varepsilon_t} K_t^{\alpha} L_t^{1-\alpha}$$

  fits our previous (Harrod-neutral) technical progress considerations, it is a bit inconvenient.
- We can get rid of $\tilde{A}^{1-\alpha}$ by normalizing $\tilde{A} = 1$.
- In the case of normality, of course, *nothing is lost* by redefining "$\varepsilon_t = (1-\alpha)\varepsilon_t$".
- This leads to: $Y_t = z_t K_t^{\alpha} L_t^{1-\alpha}$, with $z_t = e^{\varepsilon_t}$; a convenient formulation often seen in the RBC literature.

A Solow Model with Stochastic Technology (Productivity) IV

- Denote for a positive variable X_t and some constant value X^* the logarithmic difference as:

$$\hat{X}_t = \ln X_t - \ln X^* = \ln \left(\frac{X_t}{X^*} \right)$$
$$X_t = X^* e^{\hat{X}_t}$$

- For values close to X^*, equivalently for small values of $\hat{X}_t$, the following approximations “are useful” (in case all objects are well defined):

$$e^{\hat{X}_t} \simeq 1 + \hat{X}_t$$
$$e^{\hat{X}_t + b\hat{Y}_t} \simeq 1 + \hat{X}_t + b\hat{Y}_t$$
$$\hat{X}_t \hat{Y}_t \simeq 0$$
$$\mathbb{E}(e^{\hat{X}_t}) \simeq \mathbb{E}(\hat{X}_t) + \text{constant}$$

[Remember Jensen’s inequality]

A Solow Model with Stochastic Technology (Productivity) V

- Let us log-linearize the above nonlinear stochastic difference equation:

$$(1+n)k^*(1+\hat{k}_{t+1}) = (1-\delta)k^*(1+\hat{k}_t) + s\tilde{A}^{1-\alpha}(1+(1-\alpha)\varepsilon_t)(k^*)^\alpha(1+\alpha\hat{k}_t)$$

- Subtract the (equation for the) non-stochastic steady state:

$$(1+n)k^* = (1-\delta)k^* + s\tilde{A}^{1-\alpha}(k^*)^\alpha$$

on both sides:

$$(1+n)\hat{k}_{t+1} = (1-\delta)\hat{k}_t + s\tilde{A}^{1-\alpha}(k^*)^{\alpha-1}[\alpha\hat{k}_t + (1-\alpha)\varepsilon_t + \alpha(1-\alpha)\hat{k}_t\varepsilon_t]$$

A Solow Model with Stochastic Technology (Productivity) VI

- Reordering leads to:

$$\hat{k}_{t+1} = \frac{(1-\delta) + \alpha s \tilde{A}^{1-\alpha}(k^*)^{\alpha-1}}{1+n}\hat{k}_t + \frac{(1-\alpha)s\tilde{A}^{1-\alpha}(k^*)^{\alpha-1}}{1+n}\varepsilon_t$$

- Using $s\tilde{A}^{1-\alpha}(k^*)^{\alpha-1} = (n+\delta)$, this simplifies to:

$$\hat{k}_{t+1} = \underbrace{\frac{1-\delta+\alpha(n+\delta)}{1+n}}_{=C:\ -1<C<1}\hat{k}_t + \underbrace{\frac{(1-\alpha)(n+\delta)}{1+n}}_{=D}\varepsilon_t$$

- The log-linearized approximate solution is a stable first-order autoregression:

$$\hat{k}_{t+1} = C\hat{k}_t + u_t,$$

with $u_t = D\varepsilon_t$.

A Solow Model with Stochastic Technology (Productivity) VII

- For this equation, a unique stationary solution (on $\mathbb{Z}$) exists:

$$\hat{k}_{t+1} = \sum_{j=0}^{\infty} C^j u_{t-j} = D \sum_{j=0}^{\infty} C^j \varepsilon_{t-j}$$

- This allows, e. g., calculating the variance of the per-capita capital stock around its steady state:

$$\begin{aligned} \mathrm{Var}(\hat{k}_{t+1}) &= \mathbb{E}\left[\left(D \sum_{j=0}^{\infty} C^j \varepsilon_{t-j}\right)\left(D \sum_{j=0}^{\infty} C^j \varepsilon_{t-j}\right)\right] \\ &= D^2 \sum_{j=0}^{\infty} C^{2j} \sigma_A^2 = \frac{D^2}{1 - C^2} \sigma_A^2 \end{aligned}$$

A Solow Model with Stochastic Technology (Productivity) VIII

- The variance of capital around the non-stochastic steady state is thus:

$$\frac{(1-\alpha)^2(n+\delta)^2}{(1+n)^2-(1-\delta+\alpha(n+\delta))^2}$$

  times the variance σ_A^2 of the technology shock.

- With parameter values as mentioned in Chapter 2, e. g., $\alpha = 1/3$, $n = 0.01, 0.02$ and $\delta = 0.03, 0.04$, this proportionality factor is in the range of 0.0134 to 0.02.

- Analogously, output per capita can also be log-linearized around the deterministic steady state $y^* = \tilde{A}^{1-\alpha}(k^*)^\alpha$; using similar arguments as for $\hat{k}_t$ leads to:

$$\hat{y}_t = \alpha \hat{k}_t + (1-\alpha)\varepsilon_t$$

- Since $\hat{k}_t$ and ε_t are independent, the variance of $\hat{y}_t$ is:

$$\mathrm{Var}(\hat{y}_t) = \alpha^2 \mathrm{Var}(\hat{k}_t) + (1-\alpha)^2\sigma_A^2$$

Adding a Labor Supply Decision I

- Up to now, labor supply has not been modelled, due to the long-run nature of the considerations so far.
- For a business cycle theory, however, given the fluctuations of employment and unemployment over the business cycle, a theory where labor is inelastically supplied and fully utilized is a **non-starter**.
- We thus have to model labor supply and do this by adding disutility of work to the utility function:

$$u(c_t, 1 - \ell_t),$$

with $1 - \ell_t$ the (fraction of time, altogether one unit) worked, i. e., ℓ_t denotes labor in t – **from now on we have to distinguish between population and labor**.

Adding a Labor Supply Decision II

- A (due to its simplicity) convenient formulation is an additive logarithmic form:

$$u(c_t, 1 - \ell_t) = \ln c_t + b \ln(1 - \ell_t), \quad b > 0$$

- How does this impact the labor supply decision of households?
- Let us consider a (deterministic) static (i. e., one-period) problem:

$$\max_{c,\ell} u(c, 1 - \ell) = \max_{c,\ell} (\ln c + b \ln(1 - \ell))$$

subject to (the budget constraint):

$$c \leq w\ell,$$

with w denoting the wage (and the consumption good the numeraire).

[We already know that the budget constraint will be binding in optimum.]

Adding a Labor Supply Decision III

- The Lagrangian for this problem is given by (ignoring non-negativity of c and the constraints $0 \leq \ell \leq 1$):

$$\mathcal{L}(c, \ell, \lambda) = \ln c + b \ln(1 - \ell) + \lambda(w\ell - c)$$

- The first-order conditions are given by:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c} &: \frac{1}{c} - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial \ell} &: -\frac{b}{1-\ell} + \lambda w = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &: w\ell - c = 0 \end{aligned}$$

Adding a Labor Supply Decision IV

- The solution is given by:

  $$-\frac{b}{1-\ell}+\frac{1}{\ell}=0,$$

  which implies $\ell=\frac{1}{1+b}$, with labor supply, as expected, decreasing with increasing disutility of work: **Intratemporal trade-off**.

- In this example labor supply is independent of the wage, because of logarithmic utility and the household having no initial wealth (this implies that income and substitution effects due to changes in wages cancel each other).

  [Solve the problem with a more general utility function to see these effects at work.]

Adding a Labor Supply Decision V

- Matters are different even with logarithmic utility in the case of more than one, e. g., two periods:

  $$\max_{c_1,\ell_1,c_2,\ell_2} \left(\ln c_1 + b\ln(1-\ell_1) + \beta\left(\ln c_2 + b\ln(1-\ell_2)\right)\right)$$

  subject to:

  $$c_1 + \frac{1}{1+r}c_2 \leq w_1\ell_1 + \frac{1}{1+r}w_2\ell_2,$$

  with r denoting the real interest rate.

- The (simplified) Lagrangian is given by:

  $$\begin{aligned}\mathcal{L}(c_1,\ell_1,c_2,\ell_2,\lambda) &= \ln c_1 + b\ln(1-\ell_1) + \beta\left(\ln c_2 + b\ln(1-\ell_2)\right)\\ &\quad + \lambda\left(w_1\ell_1 + \frac{1}{1+r}w_2\ell_2 - c_1 - \frac{1}{1+r}c_2\right)\end{aligned}$$

Adding a Labor Supply Decision VI

- The first-order conditions (w/o the one w.r.t. λ) are given by:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_1} &: \frac{1}{c_1} - \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial \ell_1} &: -\frac{b}{1-\ell_1} + \lambda w_1 = 0 \\
\frac{\partial \mathcal{L}}{\partial c_2} &: \frac{\beta}{c_2} - \frac{\lambda}{1+r} = 0 \\
\frac{\partial \mathcal{L}}{\partial \ell_2} &: -\beta\frac{b}{1-\ell_2} - \frac{\lambda}{1+r} w_2 = 0
\end{aligned}$$

Adding a Labor Supply Decision VII

- These first-order conditions imply:

$$\frac{1-\ell_1}{1-\ell_2} = \frac{1}{\beta(1+r)} \frac{w_2}{w_1}$$

- This relationship shows that (relative) labor supply responds to relative wages in the two periods.
- This relationship also shows that an increase in r (or β) raises labor supply in Period 1 relative to labor supply in Period 2.
- The interest rate effect is crucial for employment fluctuations in RBC models.
- Lucas and Rapping (1969) is "the" classic study of **intertemporal substitution (effects)** in labor supply.

A Simplified Real Business Cycle Model I

- We start with a stripped down RBC model by positing the following simplifying assumptions:
  - We abstract from population growth: $N_t \equiv N = 1$ for all t, this implies that $L_t = \ell_t$.
  - It also implies that percapita quantities, e. g., $y_t = \frac{Y_t}{N_t} = Y_t$, coincide with aggregate quantities, but, when $\ell_t \neq 1$, per-capita and per-worker quantities differ.
  - We assume that depreciation is complete, i. e., $\delta = 1$; this simplifies the algebra considerably, compare Brock and Mirman (1972) or Long and Plosser (1983), since it implies that $K_{t+1} = I_t$.
- We continue to consider a closed economy without a government sector (which will be introduced later).

A Simplified Real Business Cycle Model II

- The central planner problem (decentralization "applies") is given by:

$$\max_{\{c_t\},\{k_{t+1}\},\{\ell_t\}} U_0 = \max_{\{c_t\},\{k_{t+1}\},\{\ell_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\ln c_t + b \ln(1-\ell_t)\right)$$

subject to:

$$\begin{aligned} y_t &= z_t k_t^{\alpha} \ell_t^{1-\alpha} \\ y_t &= c_t + i_t \\ k_{t+1} &= i_t \\ \ln z_t &= \bar{z} + \tilde{z}_t \\ \tilde{z}_t &= \rho \tilde{z}_{t-1} + \varepsilon_t, \quad |\rho| < 1 \end{aligned}$$

A Simplified Real Business Cycle Model III

- The (simplified) Lagrangian is given by: **[With $\mathbb{E}_0$ this is a totally different game!]**

$$\mathcal{L}(\{c_t, k_{t+1}, \ell_t, \lambda_t\}_{t=0,1,\ldots}) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\ln c_t + b \ln(1-\ell_t) + \lambda_t \left[z_t k_t^{\alpha} \ell_t^{1-\alpha} - c_t - k_{t+1}\right]\right)$$

- The first-order conditions (for $t = 0, 1, \ldots$) are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &: \mathbb{E}_0 \left(\beta^t \left(\frac{1}{c_t} - \lambda_t\right)\right) = 0 \\
\frac{\partial \mathcal{L}}{\partial k_{t+1}} &: \mathbb{E}_0 \left(-\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} \alpha z_{t+1} k_{t+1}^{\alpha-1} \ell_{t+1}^{1-\alpha}\right) = 0 \\
\frac{\partial \mathcal{L}}{\partial \ell_t} &: \mathbb{E}_0 \left(\beta^t \left(-\frac{b}{1-\ell} + \lambda_t (1-\alpha) z_t k_t^{\alpha} \ell_t^{-\alpha}\right)\right) = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_t} &: \mathbb{E}_0 \left(z_t k_t^{\alpha} \ell_t^{1-\alpha} - c_t - k_{t+1}\right) = 0
\end{aligned}$$

A Simplified Real Business Cycle Model IV

- In addition, a transversality condition has to hold: $\lim_{t\to\infty}\beta^t\mathbb{E}_0\lambda_t k_{t+1}=0$.
- The solution of the stochastic dynamic optimization problem – be it a central planner solution or a (recursive) competitive equilibrium – is a sequence of random variables, rather than a deterministic sequence.
- At each point in time, feasibility of, e. g., consumption or investment choices will depend – in this simplified model – on (the realization of) productivity z_t (and with $\delta<1$ on the productivity process), with their nature often described by policy functions or decision rules.
- Loosely speaking, the recursive nature of the decision problem utilizes $\mathbb{E}_s(x_{t+1})=\mathbb{E}_s(\mathbb{E}_t(x_{t+1}))$ for all $s\le t$ – with $\mathbb{E}_t(\cdot)$ expectation conditional upon $\mathcal{I}_t$ – and that quantities observed at time t are $\mathcal{I}_t$-measurable.
- A deeper understanding of these matters **is key for a deeper understanding of dynamic stochastic (macro)economics**, see, e. g., Stokey, Lucas with Prescott (1989) on **stochastic dynamic programming**.

A Simplified Real Business Cycle Model V

- Remember that in competitive markets factors are paid their marginal products:

$$1 + r_t = \frac{\partial f(z_t, k_t, \ell_t)}{\partial k_t} = \alpha z_t \left(\frac{k_t}{\ell_t}\right)^{\alpha-1} \qquad \text{[Using here } \delta = 1\text{]}$$

$$W_t = \frac{\partial f(z_t, k_t, \ell_t)}{\partial l_t} = (1-\alpha) z_t \left(\frac{k_t}{\ell_t}\right)^{\alpha}$$

A Simplified Real Business Cycle Model VI

- This allows arriving at the following (simplified) first-order conditions (for $t = 0, 1, \ldots$):

$$\begin{aligned} \frac{1}{c_t} &= \lambda_t \\ \lambda_t &= \beta \mathbb{E}_t \left(\frac{1}{c_{t+1}} (1 + r_{t+1}) \right) \\ \frac{b}{1-\ell_t} &= \frac{W_t}{c_t} \\ z_t k_t^{\alpha} \ell_t^{1-\alpha} - c_t - k_{t+1} &= 0 \end{aligned}$$

- Combining the first two leads to the stochastic Euler equation (for consumption):

$$\frac{1}{c_t} = \beta \mathbb{E}_t \left(\frac{1}{c_{t+1}} (1 + r_{t+1}) \right) \qquad \text{[In general with } u'(c)\text{]}$$

A Simplified Real Business Cycle Model VII

- The central planner solution “equation system”:

$$
\begin{aligned}
y_t &= c_t + i_t = c_t + k_{t+1} \\
y_t &= z_t k_t^{\alpha} \ell_t^{1-\alpha} \\
\frac{1}{c_t} &= \beta \mathbb{E}_t \left(\frac{1}{c_{t+1}} (1 + r_{t+1}) \right) \\
\frac{b}{1-\ell_t} &= \frac{W_t}{c_t} \\
1 + r_t &= \alpha z_t \left(\frac{k_t}{\ell_t} \right)^{\alpha-1} = \alpha \frac{y_t}{k_t} \\
W_t &= (1-\alpha) z_t \left(\frac{k_t}{\ell_t} \right)^{\alpha} = (1-\alpha) \frac{y_t}{\ell_t} \\
\ln z_t &= \bar{z} + \tilde{z}_t, \quad \tilde{z}_t = \rho \tilde{z}_{t-1} + \varepsilon_t
\end{aligned}
$$

A Simplified Real Business Cycle Model VIII

- A(ny) solution of the model is thus a six-dimensional stochastic process $\{y_t, c_t, k_t, \ell_t, r_t, W_t\}_{t=0,1,\dots}$. The solution depends on the univariate random exogenous productivity process $\{z_t\}_{t=0,1,\dots}$ (and a given $k_0 > 0$).
- The solution processes will be stochastically singular, being driven by a univariate input (note also: the variables are partly related by identities).
- There are two (related) differences to the analysis in, e. g., the R-C-K model – $\mathbb{E}_t(\cdot)$ in the stochastic Euler equation and random input $\{z_t\}_{t=0,1,\dots}$: Stochastic linear difference equation system with rational expectations.
- The first step is, nevertheless, solving for the deterministic steady state: Consider the model economy without technology shocks:

$$z^* \equiv e^{\bar{z}}$$

A Simplified Real Business Cycle Model IX

- The steady-state values are given by:

$$
\begin{aligned}
y^* &= \frac{1}{\alpha\beta} k^* \\
c^* &= \frac{1-\alpha\beta}{\alpha\beta} k^* \\
k^* &= (\alpha\beta z^*)^{\frac{1}{1-\alpha}} \frac{1-\alpha}{1-\alpha+b(1-\alpha\beta)} = (\alpha\beta z^*)^{\frac{1}{1-\alpha}} \ell^* \\
\ell^* &= \frac{1-\alpha}{1-\alpha+b(1-\alpha\beta)} \\
1+r^* &= \frac{1}{\beta} \\
W^* &= (1-\alpha)(\alpha\beta)^{\frac{\alpha}{1-\alpha}} (z^*)^{\frac{1}{1-\alpha}}
\end{aligned}
$$

A Simplified Real Business Cycle Model X

- The log-linearized equations are given by (using $\frac{c^*}{y^*} = 1 - \alpha\beta$, $\frac{k^*}{y^*} = \alpha\beta$, $\frac{c^*}{W^*} = \frac{1-\alpha\beta}{1-\alpha-b(1-\alpha\beta)}$):

$$
\begin{aligned}
\hat{y}_t &= (1-\alpha\beta)\hat{c}_t + \alpha\beta\hat{k}_{t+1} \\
\hat{y}_t &= \tilde{z}_t + \alpha\hat{k}_t + (1-\alpha)\hat{\ell}_t \\
\hat{c}_t &= \mathbb{E}_t\hat{c}_{t+1} - (1-\beta)\mathbb{E}_t\hat{r}_{t+1} \\
\hat{\ell}_t &= -\frac{b(1-\alpha\beta)}{1-\alpha}\hat{c}_t + \frac{b(1-\alpha\beta)}{1-\alpha}\hat{W}_t \\
\hat{r}_t &= \frac{1}{1-\beta}\hat{y}_t - \frac{1}{1-\beta}\hat{k}_t \\
\hat{W}_t &= \hat{y}_t - \hat{\ell}_t \\
\tilde{z}_t &= \rho\tilde{z}_{t-1} + \varepsilon_t
\end{aligned}
$$

A Simplified Real Business Cycle Model XI

- Defining $X_t := (\hat{y}_t, \hat{c}_t, \hat{k}_t, \hat{\ell}_t, \hat{r}_t, \hat{W}_t)'$, upon re-ordering, the above ("redundant") equation system can be written as:

$$\begin{bmatrix} 0 & 0 & \alpha\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -(1-\beta) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbb{E}_t X_{t+1} = \begin{bmatrix} 1 & -(1-\alpha\beta) & 0 & 0 & 0 & 0 \\ -1 & 0 & \alpha & 1-\alpha & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{b(1-\alpha\beta)}{1-\alpha} & 0 & 1 & 0 & -\frac{b(1-\alpha\beta)}{1-\alpha} \\ \frac{1}{1-\beta} & 0 & -\frac{1}{1-\beta} & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & -1 \end{bmatrix} X_t + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tilde{z}_t$$

$$\tilde{z}_t = \rho \tilde{z}_{t-1} + \varepsilon_t$$

- The system can effectively be described by considering the dynamic equations only, i. e., zoom in on bivariate $X_t := (\hat{k}_t, \hat{c}_t)'$; then compute the other variables as functions of X_t and $\tilde{z}_t$.

A Simplified Real Business Cycle Model XII

- This leads to $A_0 \mathbb{E}_t X_{t+1} = A_1 X_t + B_0 \tilde{z}_t$, with:

$$A_0 := \begin{bmatrix} \left[1 - \frac{1-\alpha+\boldsymbol{b}(1-\alpha\beta)}{\beta[1-\alpha+\alpha\boldsymbol{b}(1-\alpha\beta)]}\right] & \alpha\beta\left[1 + \frac{(1-\alpha)[1-\alpha+\boldsymbol{b}(1-\alpha\beta)(1+\alpha\beta)]}{1-\alpha+\alpha\boldsymbol{b}(1-\alpha\beta)}\right] \\ 1 & 0 \end{bmatrix}$$

$$A_1 := \begin{bmatrix} 0 & 1 \\ \frac{1-\alpha+\boldsymbol{b}(1-\alpha\beta)}{\beta[1-\alpha+\alpha\boldsymbol{b}(1-\alpha\beta)]} & -\frac{(1-\alpha\beta)[1-\alpha+\boldsymbol{b}(1-\alpha\beta)(1+\alpha\beta)]}{\alpha\beta[1-\alpha+\alpha\boldsymbol{b}(1-\alpha\beta)]} \end{bmatrix}$$

$$B_0 := \begin{bmatrix} \frac{\rho[1-\alpha+\boldsymbol{b}(1-\alpha\beta)]}{\alpha\beta[1-\alpha+\alpha\boldsymbol{b}(1-\alpha\beta)]} \\ \frac{1-\alpha+\boldsymbol{b}(1-\alpha\beta)}{\alpha\beta[1-\alpha+\alpha\boldsymbol{b}(1-\alpha\beta)]} \end{bmatrix}$$

- What about the parameter values: $\alpha = 1/3$? $\beta = 1/(1+r^*)$ maybe also “retrievable from data”, but what about b? [Choose to “mimic” some property of the data...**calibration**]

A Simplified Real Business Cycle Model XIII

- The starting point for deriving the solution is thus of the form:

$$\begin{aligned} A_0 \mathbb{E}_t X_{t+1} &= A_1 X_t + B_0 \tilde{z}_t \\ \tilde{z}_t &= \rho \tilde{z}_{t-1} + \varepsilon_t \end{aligned}$$

- In the case that A_0 is regular – if not see, e. g., King and Watson (1998) – this system can be rewritten as:

$$\begin{aligned} \mathbb{E}_t X_{t+1} &= A_0^{-1} A_1 X_t + A_0^{-1} B_0 \tilde{z}_t \\ &= A X_t + B \tilde{z}_t \\ \tilde{z}_t &= \rho \tilde{z}_{t-1} + \varepsilon_t \end{aligned}$$

A Simplified Real Business Cycle Model XIV

- To state something generic and (maybe) obvious: The above equation system has, for given input sequence $\{\varepsilon_t\}$, *probably* many solutions, potentially with quite different properties (stationary,...) and "interpretations".
- Studying the (complete set of) solutions of linear rational expectations systems is an important topic that definitely deserves – since we are ultimately often interested in quantitative results – **much more time than we have**.
- The topic is (potentially) taken up again in **Time Series Econometrics**.
- Some variations on the theme are considered in Blanchard and Kahn (1980), Binder and Pesaran (1997), Klein (2000), Sims (2002) or Uhlig (1999).

A Simplified Real Business Cycle Model XV

- In our example, the variable k_{t+1} is pre-determined, i. e., known at (the end of) time period t – $k_{t+1} = i_t$.
- Consumption c_t is the "forward-looking" variable that is not measurable with respect to $\mathcal{I}_t$; its optimal choice depends upon conditional expectations about variables not measurable with respect to $\mathcal{I}_t$.
- In the example considered, a unique ("bounded") rational expectations equilibrium solution exists, when one of the two eigenvalues of matrix A is unstable (larger than 1 in absolute value) and one eigenvalue is stable (smaller than 1 in absolute value).
- This depends, evidently, on the underlying structural parameters of the model: α, β and b.

A Simplified Real Business Cycle Model XVI

- Assume that we are in this *determinate* (or *saddle-point stable*) case, then:

$$A = \mathcal{O} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \mathcal{O}' = \mathcal{O}\Lambda\mathcal{O}',$$

  with $|\lambda_1| < 1$, $|\lambda_2| > 1$ and $\mathcal{O}$ on *orthonormal* ($\mathcal{O}\mathcal{O}' = \mathcal{O}'\mathcal{O} = I_2$) matrix of eigenvectors:

$$\mathcal{O} = \begin{bmatrix} o_{11} & o_{12} \\ o_{21} & o_{22} \end{bmatrix}$$

- As before, in Chapter 2, we will use this decomposition to diagonalize the (now) stochastic difference equation system.

A Simplified Real Business Cycle Model XVII

$$
\begin{aligned}
\mathbb{E}_t X_{t+1} &= A X_t + B\tilde{z}_t \\
\mathbb{E}_t X_{t+1} &= \mathcal{O}\Lambda\mathcal{O}' X_t + \mathcal{O}' B\tilde{z}_t \qquad | \times \mathcal{O}' \\
\mathbb{E}_t \mathcal{O}' X_{t+1} &= \Lambda\mathcal{O}' X_t + \tilde{B}\tilde{z}_t \\
\mathbb{E}_t \tilde{X}_{t+1} &= \Lambda\tilde{X}_t + \tilde{B}\tilde{z}_t
\end{aligned}
$$

- In the transformed system the (two) coordinates are decoupled and can be solved for separately – afterwards these solutions are combined and transformed back to original variables.

A Simplified Real Business Cycle Model XVIII

- Solving (univariate time-invariant first-order) stochastic difference equations:

$$x_t = \rho x_{t-1} + u_t,$$

  for $\{u_t\}_{t\in\mathbb{Z}}$ an (exogenous) input sequence.

- For a starting value x_0, one obtains by (backward) iteration for $t \geq 0$:

$$\begin{aligned} x_1 &= \rho x_0 + u_1 \\ x_2 &= \rho x_1 + u_2 \\ &= \rho^2 x_0 + \rho u_1 + u_2 \\ &\vdots \\ x_t &= \rho^t x_0 + \sum_{j=0}^{t-1} \rho^j u_{t-j} \end{aligned}$$

A Simplified Real Business Cycle Model XIX

- Analogously, one obtains – in the case $\rho \neq 0$ – another solution by (forward) iteration ($s \geq 0$):

$$\begin{aligned} x_{t+1} &= \rho x_t + u_{t+1} \\ x_t &= \frac{1}{\rho} x_{t+1} - \frac{1}{\rho} u_{t+1} \\ x_t &= \frac{1}{\rho^2} x_{t+2} - \frac{1}{\rho^2} u_{t+2} - \frac{1}{\rho} u_{t+1} \\ &\vdots \\ x_t &= \left(\frac{1}{\rho}\right)^s x_{t+s} - \sum_{j=1}^{s} \left(\frac{1}{\rho}\right)^j u_{t+j} \end{aligned}$$

A Simplified Real Business Cycle Model XX

- Evidently, the mathematical properties of the solutions depend on $|\rho|$, the properties of $\{u_t\}_{t\in\mathbb{Z}}$ and of x_0 or x_{t+s} (or x_{t-s}).
- Note that similarly to x_{t+s} in the forward solution, one can also consider a starting time $t-s$ for $s \geq 0$ in the backward solution:

$$x_t = \rho^s x_{t-s} + \sum_{j=0}^{s-1} \rho^j u_{t-j}$$

- To simplify the discussion, we assume that the input sequence $\{u_t\}_{t\in\mathbb{Z}}$ is White Noise with variance $\sigma_u^2 > 0$.

A Simplified Real Business Cycle Model XXI

Stationary Solution on $\mathbb{Z}$ with White Noise Input

- In the case that $|\rho| < 1$, the unique (weakly) stationary solution of the AR(1) difference equation on $\mathbb{Z}$ is given by:

$$x_t = \sum_{j=0}^{\infty} \rho^j u_{t-j}$$

- In the case that $|\rho| > 1$, the unique (weakly) stationary solution of the AR(1) difference equation on $\mathbb{Z}$ is given by:

$$x_t = -\sum_{j=1}^{\infty} \left(\frac{1}{\rho}\right)^j u_{t+j}$$

- In the case that $|\rho| = 1$, no (weakly) stationary solution on $\mathbb{Z}$ exists.

A Simplified Real Business Cycle Model XXII

- To calculate a unique backward solution exactly – in the case $|\rho| < 1$ – one needs an initial value x_0, in our example that would be the initial capital stock.
- However, the impact of x_0 decays exponentially in this case; the issue would be slightly different in case of considering x_{t-s} and $s \to \infty$.
- Analogously, for the case that $|\rho| > 1$, the exact calculation of the forward solution requires known x_{t+s} for the $s > 0$ considered.
- Here it is typically assumed that $\lim_{s\to\infty} \left(\frac{1}{\rho}\right)^s x_{t+s} = 0$; and that $\sum_{j=1}^{\infty} \left(\frac{1}{\rho}\right)^j u_{t+j}$ is well defined.
- This rules out x_t sequences that grow faster than $(1/\rho)^s$ decays: Relations to transversality conditions, exclusion of bubbles,...

A Simplified Real Business Cycle Model XXIII

- Back to our problem:

$$\mathbb{E}_t \tilde{X}_{t+1} = \Lambda \tilde{X}_t + \begin{bmatrix} \tilde{b}_{11} \\ \tilde{b}_{21} \end{bmatrix} \tilde{z}_t, \quad \tilde{z}_t = \rho \tilde{z}_{t-1} + \varepsilon_t$$

- Solve the “unstable” equation in forward direction:

$$\mathbb{E}_t \tilde{X}_{2,t+1} = \lambda_2 \mathbb{E}_t \tilde{X}_{2,t} + \tilde{b}_{21} \mathbb{E}_t \tilde{z}_t$$

to obtain:

$$\tilde{X}_{2,t} = \underbrace{\lim_{s \to \infty} \left(\frac{1}{\lambda_2} \right)^s \mathbb{E}_t \tilde{X}_{2,t+s}}_{=0} - \left(\frac{\tilde{b}_{21}}{\lambda_2} \right) \sum_{j=0}^{\infty} \left(\frac{1}{\lambda_2} \right)^j \mathbb{E}_t \tilde{z}_{t+j}$$

A Simplified Real Business Cycle Model XXIV

- In the special case considered with $\tilde{z}_t$ a (causal) autoregressive process of order one, it holds that:

$$\mathbb{E}_t \tilde{z}_{t+j} = \rho^j \tilde{z}_t$$

- This implies:

$$\begin{aligned}\tilde{X}_{2,t} &= -\left(\frac{\tilde{b}_{21}}{\lambda_2}\right) \sum_{j=0}^{\infty} \left(\frac{1}{\lambda_2}\right)^j \rho^j \tilde{z}_t \\ &= -\frac{\tilde{b}_{21}}{\lambda_2 - \rho} \tilde{z}_t\end{aligned}$$

A Simplified Real Business Cycle Model XXV

- The transformation to a diagonal system above, with:

$$\mathcal{O}' \begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix} = \tilde{X}_t$$

implies that:

$$\tilde{X}_{2,t} = o_{12}\hat{k}_t + o_{22}\hat{c}_t$$

- This implies the law of motion (policy function) of $\hat{c}_t$ as a function of $\hat{k}_t$ and $\tilde{z}_t$:

$$\begin{aligned} \hat{c}_t &= -\frac{o_{12}}{o_{22}}\hat{k}_t + \frac{1}{o_{22}}\tilde{X}_{2,t} \\ &= -\frac{o_{12}}{o_{22}}\hat{k}_t - \frac{1}{o_{22}}\frac{\tilde{b}_{21}}{\lambda_2 - \rho}\tilde{z}_t \end{aligned}$$

A Simplified Real Business Cycle Model XXVI

- Now consider the first (stable) equation:

$$\mathbb{E}_t \tilde{X}_{1,t+1} = \lambda_1 \tilde{X}_{1,t} + \tilde{b}_{11} \tilde{z}_t$$

$$\mathbb{E}_t \tilde{X}_{1,t+1} = \lambda_1 \mathbb{E}_t \tilde{X}_{1,t} + \tilde{b}_{11} \mathbb{E}_t \tilde{z}_t$$

- The solution is straightforwardly given by:

$$\begin{aligned} \tilde{X}_{1,t} &= \tilde{b}_{11} \sum_{j=0}^{\infty} \lambda_1^j \mathbb{E}_t \tilde{z}_{t-1-j} \\ &= \tilde{b}_{11} \sum_{j=0}^{\infty} \lambda_1^j \tilde{z}_{t-1-j} \end{aligned}$$

A Simplified Real Business Cycle Model XXVII

- Combining the solutions (in terms of $\{\tilde{z}_t\}$) for the two coordinates leads to:

$$\begin{pmatrix} \tilde{X}_{1,t} \\ \tilde{X}_{2,t} \end{pmatrix} = \begin{pmatrix} \tilde{b}_{11} \sum_{j=0}^{\infty} \lambda_1^j \tilde{z}_{t-1-j} \\ -\frac{\tilde{b}_{21}}{\lambda_2 - \rho} \tilde{z}_t \end{pmatrix} = \tilde{B}(L)\tilde{z}_t$$

- The solution for $X_t = (\hat{k}_t, \hat{c}_t)'$ follows immediately from:

$$\begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix} = \mathcal{O}\tilde{X}_t = \mathcal{O} \begin{pmatrix} \tilde{b}_{11} \sum_{j=0}^{\infty} \lambda_1^j \tilde{z}_{t-1-j} \\ -\frac{\tilde{b}_{21}}{\lambda_2 - \rho} \tilde{z}_t \end{pmatrix} = B(L)\tilde{z}_t,$$

  with $B(L)$ a tall (2-by-1) transfer function, reflecting the **stochastic singularity** of the model.

A Simplified Real Business Cycle Model XXVIII

- It may, however, be more convenient to describe the solution in terms of the law of motion of $\hat{k}_{t+1}$ as a function of $\hat{k}_t$ and $\tilde{z}_t$.
- This is achieved, e. g., as follows (simplification of the solution just given also works):

$$\tilde{X}_{1,t} = o_{11}\hat{k}_t + o_{21}\hat{c}_t$$

- Use the law of motion for c_t from above:

$$\begin{aligned}\tilde{X}_{1,t} &= o_{11}\hat{k}_t + o_{21}\left(-\frac{o_{12}}{o_{22}}\hat{k}_t - \frac{1}{o_{22}}\frac{\tilde{b}_{21}}{\lambda_2-\rho}\tilde{z}_t\right)\\ &= \left(o_{11} - \frac{o_{12}o_{21}}{o_{22}}\right)\hat{k}_t - \frac{o_{21}}{o_{22}}\frac{\tilde{b}_{21}}{\lambda_2-\rho}\tilde{z}_t\\ &= D_1\hat{k}_t - D_2\tilde{z}_t\end{aligned}$$

A Simplified Real Business Cycle Model XXIX

- Next, insert this expression for $\tilde{X}_{1,t}$ in the difference equation:

$$\mathbb{E}_t \left(D_1 \hat{k}_{t+1} - D_2 \tilde{z}_{t+1} \right) = \lambda_1 \left(D_1 \hat{k}_t - D_2 \tilde{z}_t \right) + \tilde{b}_{11} \tilde{z}_t$$

- Use $\mathbb{E}_t \hat{k}_{t+1} = \hat{k}_{t+1}$ and $\mathbb{E}_t \tilde{z}_{t+1} = \rho \tilde{z}_t$ to arrive at:

$$\hat{k}_{t+1} = \lambda_1 \hat{k}_t + \frac{D_2(\rho - \lambda_1) + \tilde{b}_{11}}{D_1} \tilde{z}_t$$

$$\hat{c}_t = -\frac{o_{12}}{o_{22}} \hat{k}_t - \frac{1}{o_{22}} \frac{\tilde{b}_{21}}{\lambda_2 - \rho} \tilde{z}_t$$

A Simplified Real Business Cycle Model XXX

- Altogether we have found a solution of the form:

$$\begin{pmatrix} \hat{k}_{t+1} \\ \hat{c}_t \end{pmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{pmatrix} \hat{k}_t \\ \tilde{z}_t \end{pmatrix}$$

- If one “knows” that the solution will be of this form one can insert this directly into the equation system:

$$\mathbb{E}_t X_{t+1} = A X_t + B \tilde{z}_t$$

and solve this system by an **undetermined coefficients** approach, see, e. g., McCallum (1983, 1998). [Several solutions: Choose “correct” one.]

Discrete Productivity Shocks I

- Let us now return to the issue why $\mathbb{E}_0$ is replaced by $\mathbb{E}_t$ in the (sequence of) first-order conditions.
- This is easier to see in the case of discrete productivity shocks, i. e., in the case where z_t can only assume a finite (or countable) number of values.
- Specifically, we assume – again in a stochastic neoclassical growth model (we ignore the labor supply choice for brevity, i. e., $\ell_t \equiv 1$) – that in every period productivity z_t can assume finitely many values in $\mathcal{Z} = \{z_1, \ldots, z_N\}$ with positive probabilities. [The process is often considered to be a (first-order) **Markov chain**.]
- The representative consumer (central planner) in fact needs to choose a sequence that maximizes expected utility for every possible sequence of z_t's.

Discrete Productivity Shocks II

- Denote a sequence of realizations of z_t from time zero to time T as $Z_{0:T} := (z_T, z_{T-1}, \dots, z_0)$; by definition $Z_{0:T} = (z_T, Z_{0:T-1})$.
- The probability of $Z_{0:T}$ is denoted as $\mathbb{P}(Z_{0:T})$, the set of all possible sequences $Z_{0:T}$ is denoted as $\mathcal{Z}_{0:T}$.
- The time-zero expected utility maximization problem can be "explicitly written" as:

$$\begin{aligned}\max_{\{c_t\},\{k_{t+\mathbf{1}}\}} U_0 &= \max_{\{c_t\},\{k_{t+\mathbf{1}}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \\ &= \max_{\{c_t(Z_{\mathbf{0}:t})\},\{k_{t+\mathbf{1}}(Z_{\mathbf{0}:t})\}} \sum_{t=0}^{\infty} \sum_{Z_{\mathbf{0}:t}\in\mathcal{Z}_{\mathbf{0}:t}} \beta^t \mathbb{P}(Z_{0:t}) u(c_t(Z_{0:t}))\end{aligned}$$

Discrete Productivity Shocks III

- The (sequence of) budget constraints (with $\delta = 1$) can analogously be written as:

$$\begin{aligned} c_t(Z_{0:t}) + k_{t+1}(Z_{0:t}) &= z_t f(k_t(Z_{0:t-1})) \\ k_0 &> 0 \text{ given} \end{aligned}$$

- The first-order condition w.r.t. $k_{t+1}(Z_{0:t})$ can be written as:

$$-\mathbb{P}(Z_{0:t})u'(c_t(Z_{0:t})) + \sum_{z_{t+\mathbf{1}} \in \mathcal{Z}} \beta \mathbb{P}(z_{t+1}, Z_{0:t}) u'(c_{t+1}(z_{t+1}, Z_{0:t})) \times z_{t+1} f'(k_{t+1}(Z_{0:t})) = 0$$

$$\begin{aligned} u'(c_t(Z_{0:t})) &= \sum_{z_{t+\mathbf{1}} \in \mathcal{Z}} \beta \frac{\mathbb{P}(z_{t+1}, Z_{0:t})}{\mathbb{P}(Z_{0:t})} u'(c_{t+1}(z_{t+1}, Z_{0:t})) \times z_{t+1} f'(k_{t+1}(Z_{0:t})) \\ &= \mathbb{E}_{Z_{\mathbf{0}:t}} \left[\beta u'(c_{t+1}(z_{t+1}, Z_{0:t})) \times z_{t+1} f'(k_{t+1}(Z_{0:t}))\right] \end{aligned}$$

A Baseline Real Business Cycle Model I

- The model is closely related to our discussion in Chapter 2 and we keep the same simplifications as there, e. g., only one household, i. e., $H = 1$.
- Output is produced and capital is accumulated via:

$$\begin{aligned} Y_t &= K_t^{\alpha}(A_t L_t)^{1-\alpha}, \quad 0 < \alpha < 1 \\ K_{t+1} &= K_t + I_t - \delta K_t \\ &= K_t + Y_t - C_t - {\color{red}G_t} - \delta K_t \end{aligned}$$

- Technology moves stochastically according to:

$$\begin{aligned} \ln A_t &= \bar{A} + gt + \tilde{A}_t \\ \tilde{A}_t &= \rho_A \tilde{A}_{t-1} + \varepsilon_{At}, \quad -1 < \rho_A < 1, \quad \varepsilon_{At} \sim \text{WN}(0, \sigma_A^2) \end{aligned}$$

A Baseline Real Business Cycle Model II

- Aggregate government spending G_t is considered to move randomly exogenously with government spending (demand) shocks around the "balanced growth path":

$$\ln G_t = \bar{G} + (n+g)t + \tilde{G}_t$$
$$\tilde{G}_t = \rho_G \tilde{G}_{t-1} + \varepsilon_{Gt}, \quad -1 < \rho_G < 1, \quad \varepsilon_{Gt} \sim \text{WN}(0, \sigma_G^2)$$

- The processes $\{\varepsilon_{At}\}$ and $\{\varepsilon_{Gt}\}$ are mutually independent.
- Population N_t (now not necessarily equal to labor L_t) grows at the exogenous rate n:

$$N_{t+1} = (1+n)N_t, \; N_0 > 0, \; n \geq 0$$

- In a perfect competition setting, with "many" households or individuals and firms, the central planner solution coincides with the decentralized equilibrium.

A Baseline Real Business Cycle Model III

- The central planner maximizes (using the following notation for per-capita quantities: $c_t = C_t/N_t$, $\ell_t = L_t/N_t$):

$$\max_{\{c_t\},\{K_{t+1}\},\{\ell_t\}} U_0 = \max_{\{c_t\},\{K_{t+1}\},\{\ell_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (1+n)^t \left(\ln c_t + b \ln(1-\ell_t)\right)$$

subject to:

$$\begin{aligned} Y_t &= K_t^{\alpha}(A_t L_t)^{1-\alpha} \\ Y_t &= C_t + I_t + G_t \\ K_{t+1} &= (1-\delta)K_t + I_t \\ \ln A_t &= \bar{A} + gt + \tilde{A}_t, \quad \tilde{A}_t = \rho_A \tilde{A}_{t-1} + \varepsilon_{At} \\ \ln G_t &= \bar{G} + (n+g)t + \tilde{G}_t, \quad \tilde{G}_t = \rho_G \tilde{G}_{t-1} + \varepsilon_{Gt} \end{aligned}$$

[Plus parameter restrictions, starting values,...]

A Baseline Real Business Cycle Model

The Effects of a Technology Shock I

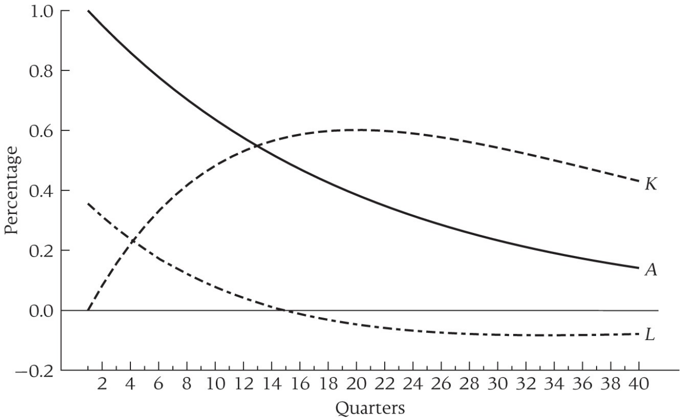

Figure 2: The effects of a 1% technology shock on the paths of technology, capital, and labor

A Baseline Real Business Cycle Model

The Effects of a Technology Shock II

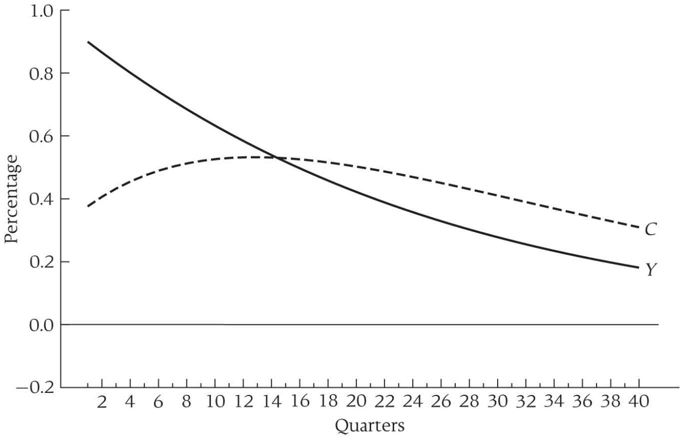

Figure 3: The effects of a 1% technology shock on the paths of output and consumption

A Baseline Real Business Cycle Model

The Effects of a Technology Shock III

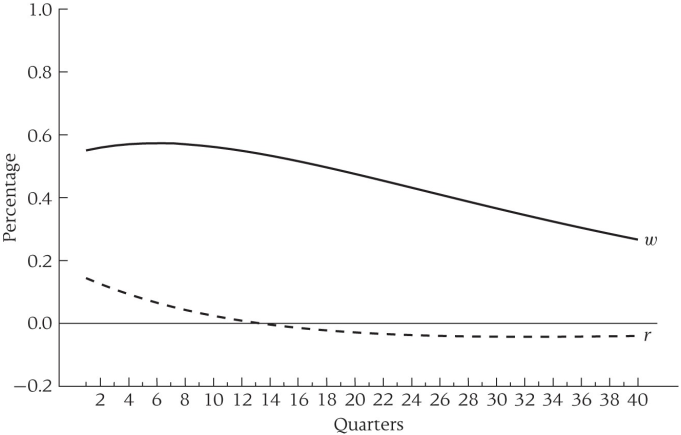

Figure 4: The effects of a 1% technology shock on the paths of the wage and the interest rate

A Baseline Real Business Cycle Model

The Effects of a Government Spending Shock I

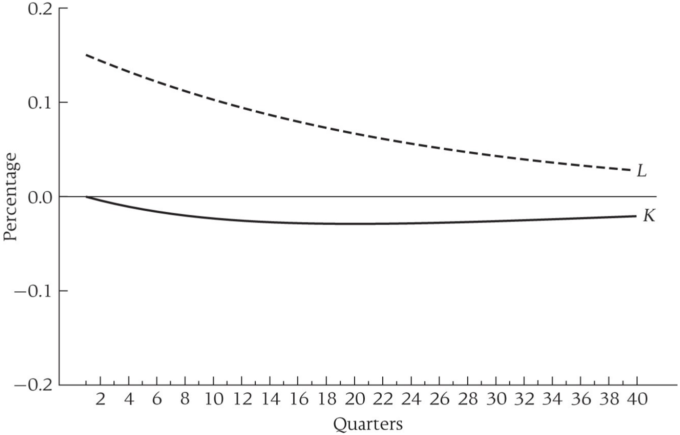

Figure 5: The effects of a 1% government-purchases shock on the paths of capital and labor

A Baseline Real Business Cycle Model

The Effects of a Government Spending Shock II

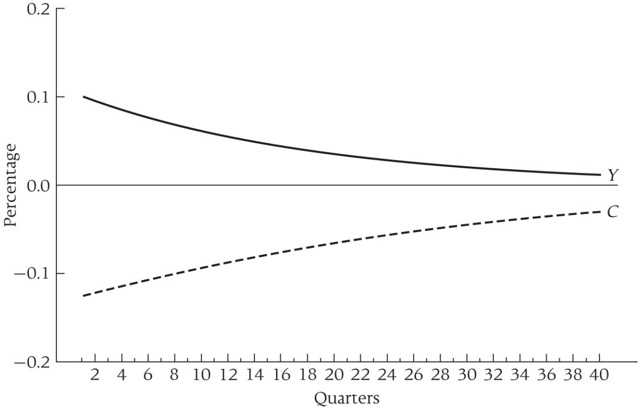

Figure 6: The effects of a 1% government-purchases shock on the paths of output and consumption

A Baseline Real Business Cycle Model

The Effects of a Government Spending Shock III

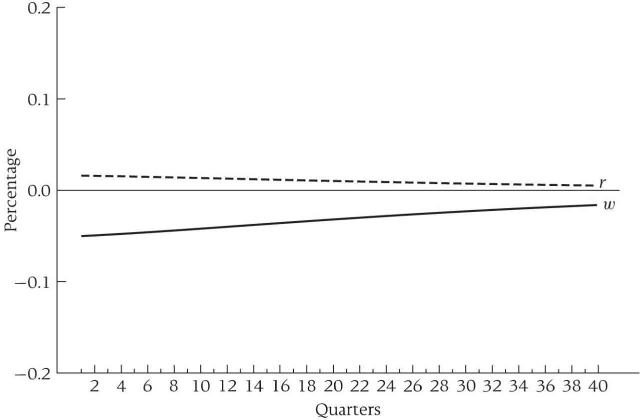

Figure 7: The effects of a 1% government-purchases shock on the paths of the wage and the interest rate

Next Steps

- A discussion of (the tools used to solve) RBC models is incomplete without solving a model with dynamic programming, using, e. g., value function iteration.
- The more frictions included, the “less likely” (if not impossible) it will be that the (or maybe a – in the case of multiple equilibria) decentralized equilibrium coincides with a central planner optimum.
- It is key that one is able to put the models to (i) the computer and (ii) to the data – loosely speaking (in the case of detrended data) an RBC model is supposed to replicate the second moment properties of the data as a function of structural shocks.
- The models discussed so far are too simplistic to capture important aspects of the (joint) behavior of (many) macroeconomic variables.
- Importantly, however, the dynamic stochastic general equilibrium approach to macroeconomics is a dominant modelling framework – in a quickly evolving literature (heterogeneous agents, multi-country models, continuous time models,...)