EC2: Macroeconomics
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This is work in progress – please check regularly for updates!

Chapter 2

The Ramsey-Cass-Koopmans Model

Setting and Assumptions

- The model – discussed in related format by Ramsey (1928), Cass (1965) and Koopmans (1965) – remedies one of the simplifying assumptions of the Solow model: The constant saving rate s.
- Instead, the model now has a form of **micro-foundation** in that a *representative* household (i. e., a large number of identical households) performs an intertemporally optimal consumption-investment decision.
- The consumption-saving decision rests upon an underlying dynamic utility maximization problem that leads to an optimally chosen, in general not time-invariant, saving rate.
- The fact that the model is based on utility functions of households also implies that welfare analysis can be performed.
- In addition to a representative household, there is also a representative firm (i. e., a large number of identical firms).
- Effectively, the main difference to the Solow model is really the utility maximization problem of the representative household.

Setting and Assumptions I

Firms and Technology

- There is a large number of identical firms that all have access to the production function $Y_t = F(K_t, A_t L_t)$ with the same assumptions as in the Solow model.
- The firms hire workers and rent capital in competitive markets, the input A_t is freely available.
- The output markets are also competitive.
- The objective function of firms is to maximize profits, the profits accrue to the owners, the households.
- Constant returns to scale, a common production function and common factor prices imply that aggregate output behaves like being produced by one big firm (using aggregate inputs).

Setting and Assumptions II

Firms and Technology

- We consider, as in Chapter 1, the production function in intensive form, i. e., we consider $f(k_t) = F(\frac{K_t}{A_t L_t}, 1)$.
- In a competitive equilibrium, it holds that $r_t = f'(k_t) - \delta$, where δ is the depreciation rate.
- For the real wage, it holds analogously in competitive equilibrium that:

$$W_t = \frac{\partial F(K_t, A_t L_t)}{\partial L_t} = A_t \frac{\partial F(K_t, A_t L_t)}{\partial A_t L_t}$$
$$= A_t \left[f(k_t) - k_t f'(k_t) \right]$$

- This implies that the real wage per unit of *effective labor*, $w_t = \frac{W_t}{A_t}$ is equal to:

$$w_t = f(k_t) - f'(k_t) k_t$$

Setting and Assumptions I

Households

- The economy is inhabited by a large number, H, of identical households: No analysis of inequality; w.l.o.g. consider $H = 1$. [A lot of macro assumes "a mass of individuals of size one".]
- The size of the household(s) grows at rate n; w.l.o.g. set $L_0 = 1$.
- Each member of a household – thus "per capita" equals "per worker" – supplies one unit of labor at every point in time: No labor-leisure choice. [$N_t = L_t$]
- Initial capital holdings $\frac{K_0}{H}$, with $H = 1$ thus simply K_0.
- Household(s) receive(s) labor and capital income (potentially profits) and make(s) a consumption-saving decision at each point in time to maximize lifetime utility.

Setting and Assumptions II

Households

- The evolution of household wealth from period t to $t+1$ is given by:

$$B_{t+1} = (1 + r_t)B_t + W_t L_t - C_t L_t,$$

  where B_t denotes household wealth at the beginning of period t.
- In this closed economy without government the only way of saving is to hold capital, thus for all t it holds that $K_t = B_t$.
- The initial capital stock K_0 is, as mentioned above, owned by the household(s), i. e., $B_0 = K_0$.
- Using this and going to quantities per effective unit of labor, with $k_t = \frac{K_t}{A_t L_t}$, $w_t = \frac{W_t}{A_t}$ and $c_t = \frac{C_t}{A_t}$ [Attention: W_t and C_t denote per capita/worker quantities here] we arrive at:

$$k_{t+1} = \frac{1}{1+n}\left[(1 + r_t)k_t + w_t - c_t\right]$$

Setting and Assumptions III

Households

- In a competitive economy, the household(s) take the time paths of r_t and w_t as given ("price takers").
- In a competitive equilibrium (this is a "nice economy" where indeed a competitive equilibrium exists), both factors are paid their marginal products:

$$
\begin{aligned}
k_{t+1} &= \frac{1}{1+g+n}\left[(1+f'(k_t)-\delta)k_t + f(k_t) - f'(k_t)k_t - c_t\right] \\
&= \frac{1}{1+g+n}\left[f(k_t) + (1-\delta)k_t - c_t\right]
\end{aligned}
$$

Setting and Assumptions IV

Households

- We already know this relationship that describes asset accumulation from the Solow model [Attention: In the Solow model discussion C_t denotes aggregate consumption]:

$$\begin{aligned} K_{t+1} &= (1-\delta)K_t + sY_t \\ &= (1-\delta)K_t + (F(K_t, L_t) - C_t) \\ k_{t+1} &= \frac{1}{1+g+n}\left[f(k_t) + (1-\delta)k_t - c_t\right] \end{aligned}$$

- We have now derived this relationship as a (key) element of a competitive equilibrium and from the distribution side.

Setting and Assumptions V

Households

- The household's objective function is given by:

$$U_0 = \sum_{t=0}^{\infty} \beta^t (1+n)^t u(C_t),$$

with C_t (in this chapter) per-capita consumption and:

- The discount factor $0 < \beta < 1$.
- The constant relative risk aversion function:

$$u(C_t) = \begin{cases} \frac{C_t^{1-\theta}}{1-\theta} & \text{if } \theta > 0, \ \theta \neq 1, \\ \ln C_t & \text{if } \theta = 1 \end{cases}$$

- Relative risk aversion is defined as $-C \frac{\frac{d^2 u(C)}{dC^2}}{\frac{du(C)}{dC}}$ and is equal to θ. [Verify.]

The Central Planner Solution I

- Let us consider the central planner or centralized solution (first): The central planner can implement the competitive equilibrium by setting prices appropriately.
- In the Solow model, a steady state in per-capita quantities only exists in the case without technical progress: We start with this case here as well, i. e., $A_t \equiv 1$.
- **In this case**, with $g = 0$, it holds that $c_t = C_t$ and $w_t = W_t$, so quantities per effective unit of labor are simply quantities per capita.
- Furthermore, to gain understanding about the type of problem to be solved we first consider a finite-horizon problem – with $t = 0, \ldots, T$.
- Thus, we consider the following optimization problem (including all constraints, some of which we already "know" to be non-binding due to Inada conditions and non satiation).

The Central Planner Solution II

$$\max_{\{C_t\}_{t=\mathbf{0},\dots,T}} U_0 = \sum_{t=0}^{T} \beta^t (1+n)^t u(C_t)$$

subject to:

$$\begin{aligned} k_0 &> 0 \text{ given} \\ k_{t+1} &= \frac{1}{1+0+n}\left[f(k_t) + (1-\delta)k_t - C_t\right], \ t = 0, \dots, T \\ k_{t+1} &\geq 0, \ t = 0, \dots, T \\ C_t &\geq 0, \ t = 0, \dots, T \end{aligned}$$

- This is a constrained maximization problem with mixed equality and inequality constraints: What about solution(s) and how to find them?

The Kuhn-Tucker Conditions I

- The problem to be solved is a special case of maximizing continuously differentiable (scalar-valued) function $f(x, y)$ with $x_i \geq 0$, $i = 1, \ldots, m$ and $y_j \in \mathbb{R}$, $j = 1, \ldots, n$.
  [Consult your math lecture notes for more (technical) details...]
- Subject to the constraints $f_k(x, y) = 0, \ k = 1, \ldots, K$ and $g_h(x, y) \geq 0, \ h = 1, \ldots, H$.
- Define the Lagrangian function:

$$\mathcal{L}(x, y, \lambda, \mu) = f(x, y) + \sum_{k=1}^{K} \lambda_k f_k(x, y) + \sum_{h=1}^{H} \mu_h g_h(x, y)$$

- It is convenient to use vector notation, e. g.,:

$$\frac{\partial \mathcal{L}}{\partial x} = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial x_1} \\ \vdots \\ \frac{\partial \mathcal{L}}{\partial x_m} \end{bmatrix}$$

The Kuhn-Tucker Conditions II

- The following first-order conditions are necessary conditions for a constrained maximum:

$$\frac{\partial \mathcal{L}}{\partial x} \leq 0, \frac{\partial \mathcal{L}}{\partial y} = 0,$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0, \frac{\partial \mathcal{L}}{\partial \mu} \geq 0,$$
$$x_i \frac{\partial \mathcal{L}}{\partial x_i} = 0, \mu_h \frac{\partial \mathcal{L}}{\partial \mu_h} = 0,$$
$$x_i \geq 0, \mu_h \geq 0$$

- Sufficiency: In the case that the objective function and the constraints are concave, the conditions are sufficient for a local maximum; strict concavity gives a unique (global) maximum. [Second-order conditions...]
- There is a well-developed theory concerning sufficient conditions, similar to the corresponding discussion in unconstrained optimization problems.

The Central Planner Solution Using Kuhn-Tucker I

- Back to our problem:
  - $x = [C_0, C_1, \ldots, C_T, k_1, \ldots, k_T, k_{T+1}]'$ and $y = \emptyset$
  - There are in total $T + 1$ equality constraints "$f_k(\cdot)$", these are given by: $f(k_t) + (1 - \delta)k_t - C_t - (1 + n)k_{t+1} = 0$ for $t = 0, \ldots, T$.
  - There are no inequality constraints, i. e., no constraints "$g_h(\cdot)$" and multipliers μ_h.
- This leads to:

$$\mathcal{L}(\{C_t, k_{t+1}, \lambda_t\}_{t=0,\ldots,T}) = \sum_{t=0}^{T} \beta^t (1+n)^t u(C_t) + \sum_{t=0}^{T} \beta^t (1+n)^t \lambda_t \left[f(k_t) + (1-\delta)k_t - C_t - (1+n)k_{t+1} \right]$$

- The first-order conditions are: $\frac{\partial \mathcal{L}}{\partial x} = 0$, $\frac{\partial \mathcal{L}}{\partial \lambda} = 0$ and $x_i \frac{\partial \mathcal{L}}{\partial x_i} = 0$:
- The inclusion of $\beta^t (1+n)^t$ in the second term implies that the Lagrange multipliers λ_t are expressed in terms of current consumption: Current Value "(Hamiltonian)"

The Central Planner Solution Using Kuhn-Tucker II

$$\frac{\partial \mathcal{L}}{\partial C_t} : \beta^t (1+n)^t u'(C_t) - \beta^t (1+n)^t \lambda_t \leq 0, \quad t = 0, \ldots, T$$

$$C_t \frac{\partial \mathcal{L}}{\partial C_t} : C_t \left(\beta^t (1+n)^t u'(C_t) - \beta^t (1+n)^t \lambda_t \right) = 0, \quad t = 0, \ldots, T$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} : \beta^{t+1} (1+n)^{t+1} \lambda_{t+1} \left[f'(k_{t+1}) + (1-\delta) \right] - \beta^t (1+n)^{t+1} \lambda_t \leq 0, \quad t = 0, \ldots, T-1$$

$$\frac{\partial \mathcal{L}}{\partial k_{T+1}} : \; -\beta^T (1+n)^{T+1} \lambda_T \leq 0$$

$$k_{t+1} \frac{\partial \mathcal{L}}{\partial k_{t+1}} = 0, \quad t = 0, \ldots, T$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} : f(k_t) + (1-\delta) k_t - C_t - (1+n) k_{t+1} = 0, \quad t = 0, \ldots, T$$

The Central Planner Solution Using Kuhn-Tucker III

- Given our CRRA utility function (Inada conditions), it can be shown that $C_t > 0$ for all $t = 0, \ldots, T$ in the optimum; this implies that $k_t > 0$ for $t = 1, \ldots, T$.
- Therefore, the first-order conditions w.r.t. C_t and k_{t+1} simplify to:

$$\begin{aligned} \beta^t (1+n)^t u'(C_t) - \beta^t (1+n)^t \lambda_t &= 0, \quad t = 0, \ldots, T \\ \beta^{t+1} (1+n)^{t+1} \lambda_{t+1} \left[f'(k_{t+1}) + (1-\delta) \right] - \beta^t (1+n)^{t+1} \lambda_t &= 0, \quad t = 0, \ldots, T-1 \\ \beta^T (1+n)^T \lambda_T k_{T+1} (1+n) &= 0 \end{aligned}$$

- The first two sets of conditions can be combined to the probably most important dynamic equation in macroeconomics: The **Euler equation** (for consumption).
- The last condition is also important, in particular when considering infinite-horizon problems.

The Central Planner Solution Using Kuhn-Tucker IV

- The Euler equation:

$$\begin{aligned} u'(C_t) &= u'(C_{t+1})\beta\left[1 + f'(k_{t+1}) - \delta\right] \\ &= u'(C_{t+1})\beta\left[1 + r_{t+1}\right] \end{aligned}$$

- The Euler equation describes the optimal **intertemporal** behavior of consumption (and saving):
  - The marginal utility of consuming a marginal extra bit in period t is $u'(C_t)$.
  - If, instead, one does not consume this marginal unit, but saves it for one period, one obtains a return of $1 + r_{t+1}$ in $t + 1$.
  - Therefore, the benefit of shifting consumption to $t + 1$ is $u'(C_{t+1})\left[1 + r_{t+1}\right]$.
  - Since, this happens only in $t + 1$, this benefit has to be discounted to period t, hence the discounting with β, i. e., $u'(C_{t+1})\beta\left[1 + r_{t+1}\right]$.
  - Optimal behavior equates these marginal benefits.

The Central Planner Solution Using Kuhn-Tucker V

- The last-period constraint: $\beta^T(1+n)^T\lambda_T k_{T+1}(1+n) = 0$
- We know that $\lambda_T = u'(C_T) > 0$.
- This implies the terminal condition $k_{T+1} = 0$.
- This is not surprising: The households are better off by consuming any capital in period T rather than saving it (without use) for period $T+1$.
- We will get back to this last-period condition in the infinite-horizon case, where matters are a bit more complicated:
  - For $T \to \infty$ and $\beta(1+n) < 1$, it holds that $\beta^T(1+n)^T\lambda_T k_{T+1}$ may converge to zero even if k_{T+1} does not converge to zero.
  - A zero limit constrains the growth of k_t as $t \to \infty$.
  - Such conditions are known as transversality or No Ponzi scheme conditions.

The Central Planner Solution Using Kuhn-Tucker VI

- The (potentially) optimal solution is characterized by two nonlinear difference equations:

$$k_{t+1} = \frac{1}{1+n}\left[f(k_t) + (1-\delta)k_t - C_t\right]$$

$$u'(C_{t+1}) = u'(C_t)\frac{1}{\beta\left[1 + f'(k_{t+1}) - \delta\right]}$$

- Furthermore, there are two boundary conditions (both on k_t): k_0 is given and $k_{T+1} = 0$ for optimal solutions.
- These are pretty complicated equations in general and the first question is, of course, whether there are solutions, if so how many, and whether the two boundary conditions on k_t are enough to pin down a unique solution.
- Remember, the dynamics of the Solow model is described by only one difference equation for k_t.

The Central Planner Solution Using Kuhn-Tucker VII

Logarithmic Utility and Cobb-Douglas Production Function

$$k_{t+1} = \frac{1}{1+n}\left[k_t^{\alpha} + (1-\delta)k_t - C_t\right]$$
$$C_{t+1} = C_t\beta\left[1 + \alpha k_{t+1}^{\alpha-1} - \delta\right]$$
$$k_0 \text{ given and } k_{T+1} = 0$$

- Still complicated, what about the case $\alpha = 1$? (Which is "a bit outside" the range we have specified: $0 < \alpha < 1$.)
- The case with $y = Ak$, where we consider here $A = 1$, has become popular in the so-called endogenous growth literature, see, Romer (1986).

The Central Planner Solution Using Kuhn-Tucker VIII

- In this Ak case, the two difference equations simplify to:

$$
\begin{aligned}
k_{t+1} &= \frac{1}{1+n}\left[k_t + (1-\delta)k_t - C_t\right] \\
&= \frac{1}{1+n}\left[(2-\delta)k_t - C_t\right] \\
C_{t+1} &= C_t \beta \left(2-\delta\right) \\
k_0 & \text{ given and } k_{T+1} = 0
\end{aligned}
$$

- The above equations are a system of two linear (deterministic) difference equations, whose solution set is well understood.

- The set of solutions is "disciplined" by two boundary conditions (an initial and a terminal condition on k_t).

The Central Planner Solution Using Kuhn-Tucker IX

- The solution can be found, e. g., by first observing that the second equation implies $C_t = C_0[\beta(2-\delta)]^t$ for $t = 1, \ldots, T$ – or equivalently $C_t = C_T[\beta(2-\delta)]^{t-T}$ for $t = 0, \ldots, T-1$.
- The equation for capital can be solved both backward and forward, i. e., starting either at $t = 0$ with $k_0 > 0$ or at $T+1$ with $k_{T+1} = 0$.
- Rewrite the k_t difference equation in backward format:

$$\begin{aligned} k_{t-1} &= \frac{1+n}{2-\delta} k_t + \frac{1}{2-\delta} C_t \\ &= \frac{1+n}{2-\delta} k_t + C_0 \beta^t (2-\delta)^{t-1} \end{aligned}$$

- This equation can be solved backward, starting with $k_{T+1} = 0$, to obtain a sequence k_t that is a function of C_0, n, δ, β, T.

The Central Planner Solution Using Kuhn-Tucker X

- Starting with $k_{T+1} = 0$, recursive substitution leads to:

$$
\begin{aligned}
k_{T-j} &= \sum_{s=0}^{j} \left(\frac{1+n}{2-\delta}\right)^{T-s} \beta^{T-s}(2-\delta)^{T-1-s} C_0 \\
&= \sum_{s=0}^{j} [\beta(1+n)]^{T-s} \frac{C_0}{2-\delta}
\end{aligned}
$$

- Combining the solution with the given initial value for k_0 pins down $C_0 = C_0(k_0, n, \delta, \beta, T)$:

$$
C_0 = \frac{(1-[\beta(1+n)])(2-\delta)}{1-[\beta(1+n)]^{T+1}} k_0
$$

The Infinite-Horizon Case I

- The infinite-horizon case provides a few complexities that have to be dealt with due to working with infinite sums and limits.
- To make sure that the objective function $U_0 = \sum_{t=0}^{\infty} \beta^t (1+n)^t u(C_t)$ converges for finite utility sequences $\{u(C_t)\}$, assume $\beta(1+n) < 1$ – but $u(\cdot)$ is unbounded...
- If, however the solution(s) of the difference equations converge to a steady state, then we have (similar to the one-equation Solow model) tools available to describe the behavior of the economic system around the steady state: **(Log-)Linear Approximation Around the Steady State**.
- The first thing to do is to solve the infinite-horizon problem:

The Infinite-Horizon Case II

$$\max_{\{C_t\}_{t=\mathbf{0},\dots}} U_0 = \sum_{t=0}^{\infty} \beta^t (1+n)^t u(C_t)$$

subject to:

$$k_0 > 0 \text{ given}$$

$$k_{t+1} = \frac{1}{1+0+n}\left[f(k_t) + (1-\delta)k_t - C_t\right], \; t = 0, \dots$$

$$k_t \geq 0, \; t = 1, \dots$$

$$C_t \geq 0, \; t = 0, \dots$$

- To select a unique (non-explosive) solution, we need to impose the transversality condition (using the same notation for the multipliers as above):

$$\lim_{t\to\infty} \beta^t (1+n)^t \lambda_t k_{t+1}(1+n) = \lim_{t\to\infty} \beta^t \lambda_t K_{t+1} = 0$$

The Infinite-Horizon Case III

- Since we are interested in (interior) steady states, we do not seek solutions for which $k_t \to 0$; the transversality condition limits the growth of k_t.

- The current-value Lagrangian is now given by:

$$\mathcal{L}(\{C_t, k_{t+1}, \lambda_t\}_{t=0,1,\ldots}) = \sum_{t=0}^{\infty} \beta^t (1+n)^t \left(u(C_t) + \lambda_t \left[f(k_t) + (1-\delta)k_t - C_t - (1+n)k_{t+1}\right]\right)$$

The Infinite-Horizon Case IV

- Similarly to the finite-horizon case, we obtain the following first-order conditions w.r.t C_t and k_{t+1}.

$$\beta^t(1+n)^t u'(C_t) - \beta^t(1+n)^t\lambda_t = 0, \quad t = 0, 1, \ldots$$
$$\beta^{t+1}(1+n)^{t+1}\lambda_{t+1}\left[f'(k_{t+1}) + (1-\delta)\right] - \beta^t(1+n)^{t+1}\lambda_t = 0, \quad t = 0, 1, \ldots$$
$$\lim_{t\to\infty} \beta^t(1+n)^t\lambda_t k_{t+1}(1+n) = 0$$

- Which, again similarly to the finite-horizon case, leads to:

$$k_{t+1} = \frac{1}{1+n}\left[f(k_t) + (1-\delta)k_t - C_t\right]$$
$$u'(C_{t+1}) = u'(C_t)\frac{1}{\beta(1+r_{t+1})} = u'(C_t)\frac{1}{\beta\left[1+f'(k_{t+1})-\delta\right]}$$

The Infinite-Horizon Case V

- Why is the transversality condition known as No Ponzi scheme condition?
- The household's net wealth in t is given by (setting empty products to one):

$$K_t = K_0 \prod_{j=0}^{t-1}(1+r_j) + \sum_{j=0}^{t-1}\left(\prod_{m=j+1}^{t-1}(1+r_m)\right)(W_jL_j - C_jL_j)$$

- Discounting to period zero leads to:

$$\frac{K_t}{\prod_{j=0}^{t-1}(1+r_j)} = K_0 + \sum_{j=0}^{t-1}\left(\frac{1}{\prod_{m=0}^{j}(1+r_m)}\right)(W_jL_j - C_jL_j)$$

The Infinite-Horizon Case VI

- The household's (discounted to period zero) intertemporal budget constraint is given by:

$$K_0 + \sum_{j=0}^{\infty} \left(\frac{1}{\prod_{m=0}^{j}(1+r_m)} \right) (W_j L_j - C_j L_j) \geq 0,$$

with this inequality going to hold with equality in equilibrium in case of non-satiation.

- Combining the above two equations and taking limits shows that the intertemporal budget constraint amounts to:

$$\lim_{t \to \infty} \frac{K_t}{\prod_{j=0}^{t-1}(1+r_j)} = 0$$

The Infinite-Horizon Case VII

- Combining the first-order conditions (that deliver the Euler equation) shows that $\prod_{j=0}^{t-1}(1+r_j) = \frac{(1+r_0)\lambda_0}{\lambda_{t-1}\beta^{t-1}}$, which yields:

$$\lim_{t\to\infty} \frac{K_t}{\prod_{j=0}^{t-1}(1+r_j)} = 0$$

$$\lim_{t\to\infty} \frac{1}{(1+r_0)\lambda_0}\beta^{t-1}(1+n)^{t-1}\lambda_{t-1}k_t(1+n) = 0$$

- Up to the constant $\frac{1}{(1+r_0)\lambda_0} = \frac{1}{(1+r_0)u'(C_0)}$, fulfilling the intertemporal budget constraint thus coincides with the transversality condition posited above.

- This rules out Ponzi schemes with perpetual rolling over of debt without ever repaying.

The Infinite-Horizon Case VIII

- The two nonlinear difference equations have a unique solution; in conjunction with the starting value and the transversality condition.
- We show next that (under certain assumptions) this solution converges to a steady state (similar to the Solow model).
- Assume a steady state $[k^*, C^*]'$ of the nonlinear equations exists:

$$k^* = \frac{1}{1+n}\left[f(k^*) + (1-\delta)k^* - C^*\right]$$
$$u'(C^*) = u'(C^*)\frac{1}{\beta\left[1 + f'(k^*) - \delta\right]}$$

- The second equation yields: $f'(k^*) = \frac{1}{\beta} - (1-\delta)$ or equivalently $(1 + r^*) = \frac{1}{\beta}$.

The Infinite-Horizon Case IX

- In words: The steady-state interest rate is equal to the inverse of the discount rate – why is this not a surprise?
- Given that $f'(\cdot)$ is strictly monotonously decreasing, one obtains: $k^* = (f')^{-1}\left(\frac{1}{\beta} - (1-\delta)\right)$.
- Equipped with k^* one obtains $C^* = f(k^*) - (\delta + n)k^*$; with assumptions in place that imply $C^* > 0$.

The Infinite-Horizon Case X

Logarithmic Utility and Cobb-Douglas Example: $u(C) = \ln C$, $f(k) = k^{\alpha}$, $0 < \alpha < 1$

- $k^* = \left(\frac{\alpha\beta}{1-\beta(1-\delta)}\right)^{\frac{1}{1-\alpha}}$
- $C^* = f(k^*) - (\delta + n)k^*$, i. e.,:
- $C^* = \left(\frac{\alpha\beta}{1-\beta(1-\delta)}\right)^{\frac{\alpha}{1-\alpha}} - (\delta + n)\left(\frac{\alpha\beta}{1-\beta(1-\delta)}\right)^{\frac{1}{1-\alpha}}$

The Infinite-Horizon Case XI

- But, does the solution (path) converge to the (unique?) steady state?

  [We need to perform stability analysis; for which it suffices (under appropriate assumptions) to consider a linearized system.]

- The two first-order conditions can be rewritten as:

$$k_{t+1} = \frac{1}{1+n}\left[f(k_t) + (1-\delta)k_t - C_t\right]$$

$$C_{t+1} = (u')^{-1}\left(\frac{u'(C_t)}{\beta\left(1 + f'\left(\frac{1}{1+n}\left[f(k_t) + (1-\delta)k_t - C_t\right]\right) - \delta\right)}\right)$$

The Infinite-Horizon Case XII

- Defining $Z_{t+1} = [k_{t+1}, C_{t+1}]'$, the above is a two-dimensional system of nonlinear, deterministic difference equations, i. e., $Z_{t+1} = F(Z_t)$.
- In general, there is no analytical solution; and without making more specific assumptions it will, in general, be hard to perform stability analysis around the steady state.
- But, prior to doing this, let us revisit our log-utility $\alpha = 1$ (Ak) example in the infinite-horizon case:

The Infinite-Horizon Case XIII

- To reset the stage, the difference equations are in the Ak $(A = 1)$ case given by:

$$\begin{bmatrix} k_{t+1} \\ C_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{2-\delta}{1+n} & -\frac{1}{1+n} \\ 0 & \beta(2-\delta) \end{bmatrix} \begin{bmatrix} k_t \\ C_t \end{bmatrix}$$

- The eigenvalues of the matrix on the right-hand side are given by: $\frac{2-\delta}{1+n}$, $\beta(2-\delta)$.
- If we think about long-run macroeconomic data, δ is maybe between 5% and 10%, n probably around 1% and the β – in steady state, which this model does not have – would be around 1/1.03 or so (using the steady-state relation $\beta = \frac{1}{1+r^*}$).
- Thus, very likely both eigenvalues are larger than one in absolute value: This is consistent with the unbounded growth of capital and consumption this case generates.

The Infinite-Horizon Case XIV

- In principle, finding the solution is quite straightforward, if we have the correct starting value: k_0 is given, but what is the corresponding "optimal" value for C_0 (to fulfill the transversality condition)?
- In the finite-horizon case, we found $C_0 = \frac{(1-[\beta(1+n)])(2-\delta)}{1-[\beta(1+n)]^{T+1}} k_0$, considering $T \to \infty$ leads to (since $\beta(1+n) < 1$ by assumption):

$$C_0 = (1 - [\beta(1+n)])\,(2-\delta)k_0$$

- This is all we need, since – denoting the above matrix with M – it holds that:

$$\begin{bmatrix} k_t \\ C_t \end{bmatrix} = M^t \begin{bmatrix} k_0 \\ C_0 \end{bmatrix} = M^t \begin{bmatrix} 1 \\ (1 - [\beta(1+n)])\,(2-\delta) \end{bmatrix} k_0$$

The Infinite-Horizon Case XV

- Since M is an upper triangular matrix, it is clear that:

$$M^t = \begin{bmatrix} \left(\frac{2-\delta}{1+n}\right)^t & m_t \\ 0 & [\beta(2-\delta)]^t \end{bmatrix}, \quad t = 1, 2, \ldots,$$

with only the upper right element a bit more complicated:

$$m_t = -\frac{1}{1+n} \frac{\left(\frac{2-\delta}{1+n}\right)^t - [\beta(2-\delta)]^t}{\frac{2-\delta}{1+n} - \beta(2-\delta)} = -\frac{\left(\frac{2-\delta}{1+n}\right)^t - [\beta(2-\delta)]^t}{(1-[\beta(1+n)])\,(2-\delta)}$$

The Infinite-Horizon Case XVI

- Inserting and simplifying leads to:

$$\begin{bmatrix} k_t \\ C_t \end{bmatrix} = \begin{bmatrix} 1 \\ (1-[\beta(1+n)])(2-\delta) \end{bmatrix} (\beta(2-\delta))^t k_0, \quad t = 1, 2, \ldots$$

- This also shows that the ratio k_t/C_t converges to $\frac{1}{(1-[\beta(1+n)])(2-\delta)}$.

The Infinite-Horizon Case XVII

- Does the solution indeed fulfill the transversality condition:

$$\lim_{t\to\infty}[\beta(1+n)]^t \lambda_t k_{t+1}(1+n) =?$$

$$\lim_{t\to\infty}[\beta(1+n)]^t \frac{1}{C_t} k_{t+1}(1+n) =?$$

$$\lim_{t\to\infty}[\beta(1+n)]^t \frac{k_{t+1}}{C_t}(1+n) =?$$

$$\lim_{t\to\infty}[\beta(1+n)]^t \frac{\beta(2-\delta)}{(1-[\beta(1+n)])(2-\delta)}(1+n) =?$$

$$\lim_{t\to\infty}\frac{[\beta(1+n)]^{t+1}}{(1-[\beta(1+n)])} = 0$$

The Infinite-Horizon Case XVIII

- The fact that the matrix M and thus its powers M^t are upper triangular in this example simplifies the calculations considerably.
- **In the case** of a diagonalizable but, e. g., not upper triangular matrix M it is more convenient to use the eigenvalue decomposition of M:

$$M = Q \Lambda Q^{-1} = Q \Lambda Q',$$

with Λ a diagonal matrix with the eigenvalues (in a defined order) and Q an orthogonal (or orthonormal) matrix of eigenvectors.

The Infinite-Horizon Case XIX

- This simplifies equation solving considerably (using here the Z_t for the vector sequence to be solved for):

$$\begin{aligned} Z_t &= M Z_{t-1} \\ Z_t &= Q\Lambda Q^{-1} Z_{t-1} \\ Q^{-1} Z_t &= \Lambda(Q^{-1} Z_{t-1}) \\ \tilde{Z}_t &= \Lambda \tilde{Z}_{t-1} \\ \tilde{Z}_t &= \Lambda^t \tilde{Z}_0 \\ Z_t &= Q\Lambda^t Q^{-1} Z_0 \end{aligned}$$

- Using this approach (by necessity) leads to the same result as just calculated above; but can be used whenever a matrix is diagonalizable (independent of its dimensions). [What are Q and Λ for the matrix M?]

The Infinite-Horizon Case XX

- Let us go back to our nonlinear problem – with the specific assumptions $u(C_t) = \ln C_t$ and $y_t = k_t^\alpha$ for $0 < \alpha < 1$.
- For this case, we have already discussed the existence of a (non-trivial) steady state $[k^*, C^*]'$ given above.
- In this case, a common solution approach is to consider a linear (sometimes a log-linear) approximate solution around the steady state.
- The nonlinear equation system (slightly transformed from above) is given by:

$$k_{t+1} = \frac{1}{1+n}\left[k_t^\alpha + (1-\delta)k_t - C_t\right]$$

$$C_{t+1} = C_t\beta\left[1 + \alpha\left(\frac{1}{1+n}[k_t^\alpha + (1-\delta)k_t - C_t]\right)^{\alpha-1} - \delta\right]$$

The Infinite-Horizon Case XXI

- The approximate (linear) solution is therefore given by:

$$\begin{bmatrix} k_{t+1} - k^* \\ C_{t+1} - C^* \end{bmatrix} = [J_M]_{|(k^*, C^*)} \begin{bmatrix} k_t - k^* \\ C_t - C^* \end{bmatrix}$$

- With the Jacobian matrix J_M, defined as (denoting the two functions above F_1 and F_2):

$$J_M = \begin{bmatrix} \left(\frac{\partial F_1}{\partial Z}\right)' \\ \left(\frac{\partial F_2}{\partial Z}\right)' \end{bmatrix} = \begin{bmatrix} \frac{\partial F_1}{\partial k_t} & \frac{\partial F_1}{\partial C_t} \\ \frac{\partial F_2}{\partial k_t} & \frac{\partial F_2}{\partial C_t} \end{bmatrix},$$

  to be evaluated at $[k^*, C^*]'$; denoting with $Z = [k, C]'$.

- This is a system of linear difference equations – as discussed above.

The Infinite-Horizon Case XXII

$$J_M = \begin{bmatrix} \frac{1}{1+n}\left[\alpha k_t^{\alpha-1} + (1-\delta)\right] & -\frac{1}{1+n} \\ \frac{\alpha(\alpha-1)\beta C_t[\alpha k_t^{\alpha-1}+(1-\delta)]}{(1+n)(\frac{1}{1+n}[k_t^{\alpha}+(1-\delta)k_t-C_t])^{2-\alpha}} & \begin{matrix} \beta[1+\alpha(\frac{1}{1+n}[k_t^{\alpha}+(1-\delta)k_t-C_t])^{\alpha-1}-\delta]- \\ \frac{1}{1+n}\alpha(\alpha-1)\beta C_t(\frac{1}{1+n}[k_t^{\alpha}+(1-\delta)k_t-C_t])^{\alpha-2} \end{matrix} \end{bmatrix}$$

- This expression needs to be evaluated at k^* and C^* and the following relationships at steady state are useful:
  - $\frac{1}{1+n}\left((k^*)^{\alpha} + (1-\delta)k^* - C^*\right) = k^*$
  - $\alpha(k^*)^{\alpha-1} + (1-\delta) = 1 + r^* = \frac{1}{\beta}$

The Infinite-Horizon Case XXIII

- Evaluating the Jacobian at the steady state leads to:

$$J_{M|Z^*} = \begin{bmatrix} \frac{1}{\beta(1+n)} & -\frac{1}{1+n} \\ \\ \frac{\alpha(\alpha-1)}{(1+n)(k^*)^{2-\alpha}} C^* & 1 - \beta \frac{\alpha(\alpha-1)}{(1+n)(k^*)^{2-\alpha}} C^* \end{bmatrix},$$

  with (as already discussed above):

  - $k^* = \left(\frac{\alpha\beta}{1-\beta(1-\delta)} \right)^{\frac{1}{1-\alpha}}$ and $C^* = (k^*)^\alpha - (\delta + n)k^*$

- Exercise: Use "realistic" parameter values to generate/simulate the linear approximation around the steady state (starting from the appropriate $[k_0, C_0]'$).

Equilibrium: Decentralization of the Central Planner Solution I

- Under the assumptions made, the discussed central planner solution path coincides with a competitive equilibrium.
- In a competitive equilibrium, with **“many”** (identical) households (rather: individuals) and firms, every actor takes the prices – the rates of return on capital r_t and the wages W_t – as given and maximizes:
    - Household(s): Discounted utility stream subject to budget constraints, no Ponzi condition and initial wealth k_0
    - Firm(s): Profits subject to technological constraint: production function
- In equilibrium (an infinite sequence), the equilibrium sequence of prices $\{r_t, W_t\}_{t\geq 0}$, leads to market clearing on all markets in all periods. [See exercises for more details.]

Adding Exogenous Technical Progress to the R-C-K Model I

- We now add exogenous technical progress of the form $A_t = (1+g)^t A_0$ for some given $A_0 > 0$, w.l.o.g. $A_0 = 1$.
- It can be expected from the discussion of the Solow model that this will imply a balanced growth path – and a steady state in per unit of effective labor rather than in per capita quantities.
- This implies that it will be *a good idea* to try setting up the maximization problem in units of effective labor – the chosen functional form of the period utility function is of great help in this respect (with $c_t = \frac{C_t}{A_t}$ and C_t in this chapter per capita consumption):

$$\begin{aligned} u(C_t) &= \frac{(A_t c_t)^{1-\theta}}{1-\theta} \\ &= (1+g)^{(1-\theta)t} {A_0}^{1-\theta} u(c_t) \end{aligned}$$

Adding Exogenous Technical Progress to the R-C-K Model II

$$\max_{\{C_t\}_{t=0,\ldots}} U_0 = \sum_{t=0}^{\infty} \beta^t (1+n)^t u(C_t)$$

$$\max_{\{c_t\}_{t=0,\ldots}} U_0 = \sum_{t=0}^{\infty} \beta^t (1+n)^t (1+g)^{(1-\theta)t} A_0^{1-\theta} u(c_t)$$

$$\max_{\{c_t\}_{t=0,\ldots}} U_0 = A_0^{1-\theta} \sum_{t=0}^{\infty} \beta^t (1+n)^t (1+g)^{(1-\theta)t} u(c_t)$$

- To avoid further mathematical complications **assume**: $\beta(1+n)(1+g)^{1-\theta} < 1$.

Adding Exogenous Technical Progress to the R-C-K Model III

- The optimization problem is thus now given by:

$$\max_{\{c_t\}_{t=0,\dots}} U_0 = \sum_{t=0}^{\infty} \beta^t (1+n)^t (1+g)^{(1-\theta)t} u(c_t)$$

subject to:

$$k_0 > 0 \text{ given}$$
$$k_{t+1} = \frac{1}{1+g+n}\left[f(k_t) + (1-\delta)k_t - c_t\right], \ t = 0, \dots$$
$$k_t \geq 0, \ t = 1, \dots$$
$$c_t \geq 0, \ t = 0, \dots$$

[Remember from Solow: The correct "growth scaling" is $\frac{1}{(1+n)(1+g)} = \frac{1}{1+g+n+gn}$].

Adding Exogenous Technical Progress to the R-C-K Model IV

- The relevant first order conditions including the transversality condition are:

$$\begin{aligned}
\beta^t(1+n)^t(1+g)^{(1-\theta)t}u'(c_t) - \beta^t(1+n)^t(1+g)^{(1-\theta)t}\lambda_t &= 0, \quad t = 0,1,\dots \\
\beta^{t+1}(1+n)^{t+1}(1+g)^{(1-\theta)(t+1)}\lambda_{t+1}\left[f'(k_{t+1})+(1-\delta)\right] & \\
-\beta^t(1+n)^t(1+g)^{(1-\theta)t}\lambda_t(1+g+n) &= 0, \quad t = 0,1,\dots \\
\lim_{t\to\infty}\beta^t(1+n)^t(1+g)^{(1-\theta)t}(1+g+n)\lambda_t k_{t+1} &= 0
\end{aligned}$$

- The consumption Euler equation is given by:

$$u'(c_{t+1}) = u'(c_t)\frac{1+g+n}{(1+n)(1+g)^{1-\theta}\beta[f'(k_{t+1})+(1-\delta)]}$$

Adding Exogenous Technical Progress to the R-C-K Model V

- It can be rewritten (to relate to the previous setting) as:

$$u'(c_t) = u'(c_{t+1})\beta[1 + r_{t+1}]\frac{(1+n)(1+g)^{1-\theta}}{1+g+n}$$

- **Scaling with the exact term** $(1+n)(1+g)$ **rather than** $1+g+n$ **simplifies matters**:

$$u'(c_t) = u'(c_{t+1})\beta[1 + r_{t+1}](1+g)^{-\theta}$$

- For the considered CRRA utility function it holds that:

$$u'(c) = c^{-\theta}$$

Adding Exogenous Technical Progress to the R-C-K Model VI

- The two-equation nonlinear system with exact scaling is thus given by:

$$k_{t+1} = \frac{1}{(1+n)(1+g)}\left[f(k_t) + (1-\delta)k_t - c_t\right]$$

$$c_{t+1}^{-\theta} = c_t^{-\theta}\frac{1}{\beta[f'(\underbrace{\frac{1}{(1+n)(1+g)}[f(k_t)+(1-\delta)k_t - c_t]}_{=k_{t+1}}) + (1-\delta)]} \times (1+g)^{\theta}$$

- It is convenient to rewrite the second equation in the Cobb-Douglas case as:

$$c_{t+1} = c_t(1+g)^{-1}\beta^{\frac{1}{\theta}}\left[\alpha\left(\frac{1}{(1+n)(1+g)}[k_t^{\alpha} + (1-\delta)k_t - c_t]\right)^{\alpha-1} + (1-\delta)\right]^{\frac{1}{\theta}}$$

Adding Exogenous Technical Progress to the R-C-K Model VII

- The next task is to find a (unique?) steady state:

$$1 + r^* = f'(k^*) + (1 - \delta) = \left(\frac{1 + g}{\beta} \right)^{\theta}$$

- In the Cobb-Douglas case this allows to calculate k^* explicitly:

$$k^* = \left(\frac{1}{\alpha} \left[\left(\frac{1 + g}{\beta} \right)^{\theta} - (1 - \delta) \right] \right)^{\frac{1}{\alpha - 1}}$$
$$c^* = (k^*)^{\alpha} - \left[(1 + n)(1 + g) - (1 - \delta) \right] k^*$$

Adding Exogenous Technical Progress to the R-C-K Model VIII

$$J_M = \begin{bmatrix} \frac{1}{(1+n)(1+g)}\left[\alpha k_t^{\alpha-1} + (1-\delta)\right] & -\frac{1}{(1+n)(1+g)} \\ \frac{\alpha(\alpha-1)\beta^{1/\theta}\mathcal{D}^{1/\theta-1}}{(1+n)(1+g)^2} \times & \frac{\beta^{1/\theta}\mathcal{D}^{1/\theta}}{1+g} - \\ \left(\frac{1}{(1+n)(1+g)}[k_t^{\alpha} + (1-\delta)k_t - c_t]\right)^{\alpha-2} \times & \frac{\alpha(\alpha-1)\beta^{1/\theta}\mathcal{D}^{1/\theta-1}}{(1+n)(1+g)^2} \times \\ [\alpha k_t^{\alpha-1} + (1-\delta)]c_t & \left(\frac{1}{(1+n)(1+g)}[k_t^{\alpha} + (1-\delta)k_t - c_t]\right)^{\alpha-2} c_t \end{bmatrix}$$

with $\mathcal{D} := \alpha\left(\frac{1}{(1+n)(1+g)}[k_t^{\alpha} + (1-\delta)k_t - c_t]\right)^{\alpha-1} + (1-\delta)$

Adding Exogenous Technical Progress to the R-C-K Model IX

- As in the previous example, this expression can be simplified when evaluated at the steady state, where it holds that:

$$\begin{aligned} \alpha(k^*)^{\alpha-1} + (1-\delta) &= 1 + r^* \\ \mathcal{D} &= 1 + r^* \\ 1 + r^* &= \left(\frac{1+g}{\beta}\right)^{\theta} \end{aligned}$$

$$J_{M|Z^*} = \begin{bmatrix} \frac{(1+g)^{\theta-1}}{\beta^{\theta}(1+n)} & -\frac{1}{(1+n)(1+g)} \\ \frac{\alpha(\alpha-1)\beta^{1/\theta-1}}{(1+n)(1+g)(k^*)^{2-\alpha}}c^* & \beta^{1/\theta-1} - \beta^{1/\theta-1+\theta}\frac{\alpha(\alpha-1)}{(1+n)(1+g)^{1+\theta}(k^*)^{2-\alpha}}c^* \end{bmatrix}$$

Adding Exogenous Technical Progress to the R-C-K Model X

- Given the linearized behavior of $[k_t, c_t]'$ over time, the behavior of per capita (or aggregate) quantities follows from the definition of the variables in conjunction with the exogenous growth rates of technical progress and population, e. g., $C_t = A_t c_t$ and similar for the other variables.
- The behavior of the (solution of the) R-C-K model is altogether (and in particular on the BGP) very similar to the behavior of the Solow model.