

EC2: Macroeconomics

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This is work in progress – please check regularly for updates!

Chapter 1

The Solow Growth Model

Background

- **Solow (1956)** – see also **Swan (1956)** – sets out to formulate a model of economic growth compatible with the **six Kaldor (1961) stylized facts**:
 1. Labor productivity has grown at a sustained rate.
 2. Capital per worker has also grown at a sustained rate.
 3. The real interest rate, or return on capital, has been stable.
 4. The ratio of capital to output has also been stable.
 5. Capital and labor have captured stable shares of national income.
 6. Among the fast growing countries of the world, there is an appreciable variation in the rate of growth “of the order of 2–5 percent.”
- Solow built on earlier work by **Ramsey (1928)** and got the Nobel Prize in Economic Sciences in 1987 for his important contributions to theories of economic growth.
- The model is also often referred to as **Solow-Swan** model.

Setting and Assumptions I

Remark

- The discussion on economic growth in the book by **Romer** is (partly) performed using **continuous time** rather than **discrete time**, e. g., the discussion of the Solow model as well as the subsequent **micro-founded** growth models.
- In line with the rather common practice of using discrete time in (most of) the business cycle literature, we discuss the considered growth models in discrete time in class.
- A discrete-time formulation is also more directly amenable to empirical analysis of growth (compare, e. g., Romer, Chapter 5).
- In the exercises, however, we will also look at continuous-time model versions (it is important to gain familiarity with both modelling variants).

Setting and Assumptions II

- Output Y is produced using the inputs capital K , labor L and technology or effectiveness A , i. e., $Y_t = F(A_t, K_t, L_t)$.
- Production takes place with a **neoclassical production function** with **constant returns to scale** (in capital and labor) satisfying the **Inada (1963)** conditions (exemplified below).
- Technological progress is **Harrod-neutral (labor-augmenting)**, i. e., $F(A, K, L) = F(K, AL)$.
- Closed economy without government: $Y_t = C_t + I_t$, with consumption C_t and investment I_t .
- Capital depreciates at a constant rate δ , therefore $K_{t+1} = (1 - \delta)K_t + I_t$.
- There is no (so-called) **micro-foundation** (no utility function...), rather fixed shares of output are consumed and saved, i. e., there is an exogenous, constant saving rate s .

Setting and Assumptions III

- Exogenous growth rates of both labor (population) and technological progress:

$$A_{t+1} = (1 + g)A_t, \quad A_0 > 0, \quad g \geq 0$$

$$L_{t+1} = (1 + n)L_t, \quad L_0 > 0, \quad n \geq 0$$

- Consider, e. g., a **Cobb-Douglas** production function with constant returns to scale:

$$Y_t = F(K_t, A_t L_t) = K_t^\alpha (A_t L_t)^{1-\alpha}, \quad 0 < \alpha < 1$$

$$\frac{Y_t}{A_t L_t} = F\left(\frac{K_t}{A_t L_t}, 1\right) = \left(\frac{K_t}{A_t L_t}\right)^\alpha$$

$$y_t = f(k_t) = k_t^\alpha$$

- In the above example the **intensive-form** production function $y_t = k_t^\alpha$ fulfills:

- $f(0) = 0$, $f'(k_t) = \alpha k_t^{\alpha-1} > 0$, $f''(k_t) = \alpha(\alpha - 1)k_t^{\alpha-2} < 0$
- $\lim_{k \rightarrow 0} f'(k) = \infty$, $\lim_{k \rightarrow \infty} f'(k) = 0$

Solution I

- The dynamic behavior of the model economy is governed by the behavior of the capital stock per unit of effective labor:

$$\begin{aligned}
 K_{t+1} &= K_t(1 - \delta) + sY_t \\
 \frac{K_{t+1}}{A_{t+1}L_{t+1}} \frac{A_{t+1}L_{t+1}}{A_tL_t} &= \frac{K_t}{A_tL_t}(1 - \delta) + s \frac{Y_t}{A_tL_t} \\
 k_{t+1}(1 + g)(1 + n) &= k_t(1 - \delta) + sk_t^\alpha
 \end{aligned}$$

- The quantity $sf(k_t) = sk_t^\alpha$ is gross investment per unit of effective labor.
- Assume that $k_{t+1} = k_t = k^*$ is constant:

$$\begin{aligned}
 k^*((1 + g)(1 + n) - (1 - \delta)) &= s(k^*)^\alpha \\
 k^*(g + n + \underbrace{\delta + g \times n}_{\simeq 0}) &= s(k^*)^\alpha = sf(k^*)
 \end{aligned}$$

Solution II

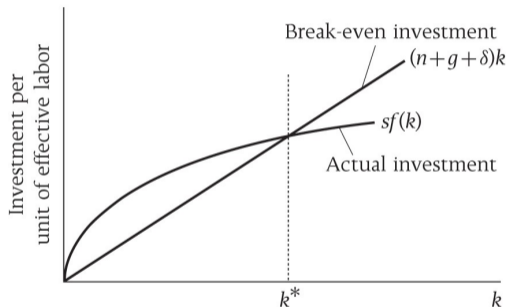


Figure 1: Actual and break-even investment

- A constant capital stock, i. e. $k_{t+1} = k_t$, requires – when setting $gn = 0$ – that:

$$sf(k^*) = (s(k^*))^\alpha = (g + n + \delta)k^*$$

- $(g + n + \delta)k$ is therefore called **break-even investment**.

Solution III

- For $k_0 > 0$, because $0 < \alpha < 1$, the (solution sequence of the) above difference equation in fact converges to a unique **steady state**:

$$k^* = \left(\frac{s}{g + n + \delta + gn} \right)^{\frac{1}{1-\alpha}}$$

- For the evolution of output and output per capita (worker) on the **balanced growth path** this implies:

$$Y_t^* = \left(\frac{s}{g + n + \delta + gn} \right)^{\frac{\alpha}{1-\alpha}} A_0 (1 + g)^t L_t$$
$$\frac{Y_t^*}{L_t} = \left(\frac{s}{g + n + \delta + gn} \right)^{\frac{\alpha}{1-\alpha}} A_0 (1 + g)^t$$

Solution IV

	Variable	Gr. rate
L	Labor	n
A	Knowledge	g
K	Capital stock	$n + g$
Y	Output	$n + g$
$\frac{K}{L}$	Capital per worker	g
$\frac{Y}{L}$	Output per worker	g
$k = \frac{K}{AL}$	Capital per eff. worker	0
$y = \frac{Y}{AL}$	Output per eff. worker	0

Table 1: Properties of the BGP

- On the balanced growth path (BGP) it holds that:
 - Capital and output per effective worker are constant.
 - Capital and output per worker grow at the rate of technological progress.
 - Capital and output grow at a rate equal to the sum of the growth rates of progress and population.

Some Empirical Evidence on Growth

	Rate of Growth of Output per Worker (%) 1985–2014	Rate of Technological Progress (%) 1985–2013
France	1.3	1.4
Japan	1.6	1.7
United Kingdom	1.9	1.4
United States	1.7	1.4
Average	1.6	1.5

Table 2: Average annual rates of growth of output per worker and technological progress in four industrialized economies (*Source*: OECD)

The Effects on Investment and k^*

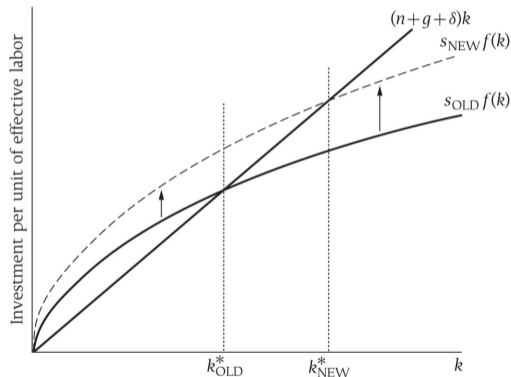


Figure 2: The effects of an increase in the saving rate on investment

- The saving rate s is (i) important and (ii) very likely to be affected by policy.
- We start from a Solow economy on the BGP and consider a **permanent** increase in s .
- This implies that k^* rises – actual investment above break-even investment.
- Monotone rise of k until new steady-state value k^* is reached.
- See the left panel of the figure on the following slide.

A Graphical Display of the Impacts

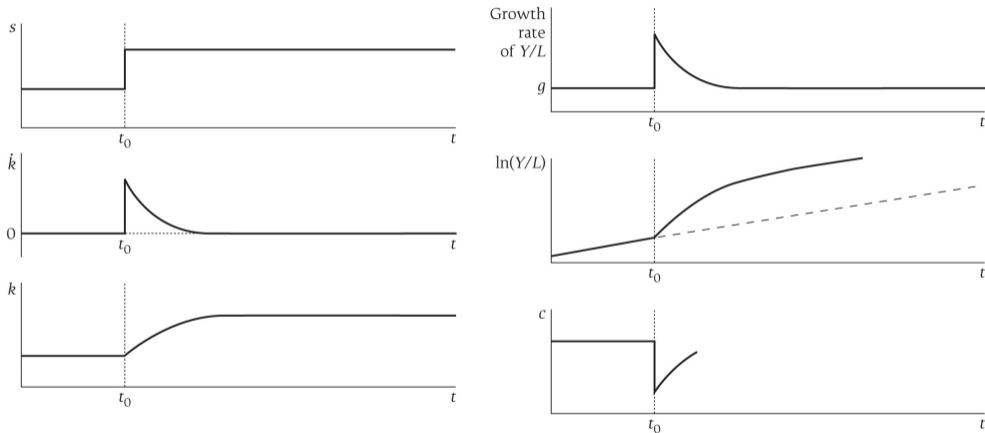


Figure 3: The effects of an increase in the saving rate

The Impacts on Output and Consumption

- The dynamics of output per worker (per capita) $\frac{Y}{L}$:
 - On the BGP, with k constant $\frac{Y}{L}$ grows at rate g .
 - After the increase in s , $\frac{Y}{L}$ grows faster – g plus (fraction) of increase in output because of k growing: $\frac{Y}{L} = Af(k)$.
 - **Permanent** increase in saving rate s implies **temporary** increase in output per worker: **Level effect**

- The impact on consumption per unit of effective labor (of primary relevance for **welfare analysis**):
 - Since $c = (1 - s)f(k)$, an increase in s – for given k – implies a downwards “jump” in c .
 - After this initial drop, c starts growing again – however, it is unclear whether it converges to a higher or lower level than the initial value.

The Impacts on Output and Consumption

The Golden Rule

- In steady state (on the BGP) it holds that:

$$c^* = f(k^*) - (n + g + \delta)k^*,$$

with $k^* = k^*(s, g, n, \delta)$.

- The impact of a (marginal) change in s on c^* is given by:

$$\frac{\partial c^*}{\partial s} = [f'(k^*(s, g, n, \delta)) - (g + n + \delta)] \underbrace{\frac{\partial k^*(s, g, n, \delta)}{\partial s}}_{>0}$$

- The sign thus depends on the first term, i. e., whether $f'(k^*(s, g, n, \delta)) \stackrel{?}{\gtrless} (g + n + \delta)$
- The saving rate that maximizes c^* – where necessarily $f'(k^*(s, g, n, \delta)) = (g + n + \delta)$ holds – is known as **golden rule**. [How big would the corresponding marginal product be approximately?]

Output, Investment and Consumption on the BGP

Per Unit of Effective Labor

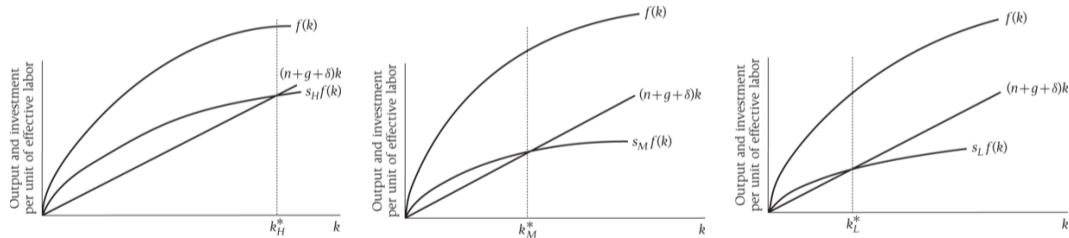


Figure 4: Output, investment, and consumption on the BGP

The Effect on Output per Unit of Effective Labor in the Long Run I

An Example of Using Approximations to Gauge Effects

$$\frac{\partial y^*}{\partial s} = f'(k^*(s, g, n, \delta)) \frac{\partial k^*(s, g, n, \delta)}{\partial s}$$

- Thus, to find $\frac{\partial y^*}{\partial s}$, we need to find $\frac{\partial k^*}{\partial s}$.
- k^* satisfies $sf(k^*(s, g, n, \delta)) = (g + n + \delta)k^*(s, g, n, \delta)$, therefore *implicit* differentiation yields:

$$sf'(k^*) \frac{\partial k^*}{\partial s} + f(k^*) = (g + n + \delta) \frac{\partial k^*}{\partial s}$$

- This leads to:

$$\frac{\partial k^*(s, g, n, \delta)}{\partial s} = \frac{f(k^*(s, g, n, \delta))}{\underbrace{(g + n + \delta) - sf'(k^*(s, g, n, \delta))}_{>0 \text{ (Why?)}}}$$

The Effect on Output per Unit of Effective Labor in the Long Run II

An Example of Using Approximations to Gauge Effects

- Inserting now leads to:

$$\frac{\partial y^*(s, g, n, \delta)}{\partial s} = \frac{f'(k^*(s, g, n, \delta))f(k^*(s, g, n, \delta))}{(g + n + \delta) - sf'(k^*(s, g, n, \delta))}$$

- Remember for subsequent transformations that in steady state $sf(k^*) = (g + n + \delta)k^*$.
- For gauging marginal effects, it is convenient to consider [elasticities](#):

$$\begin{aligned} \frac{s}{y^*} \frac{\partial y^*}{\partial s} &= \frac{s}{f(k^*)} \frac{f'(k^*)f(k^*)}{(g + n + \delta) - sf'(k^*)} \\ &= \frac{(g + n + \delta)k^* f'(k^*)}{f(k^*) \left[(g + n + \delta) - \frac{(g+n+\delta)k^* f'(k^*)}{f(k^*)} \right]} \\ &= \frac{k^* f'(k^*)/f(k^*)}{1 - [k^* f'(k^*)/f(k^*)]} \end{aligned}$$

The Effect on Output per Unit of Effective Labor in the Long Run III

An Example of Using Approximations to Gauge Effects

- Note that (by definition of elasticities) $\alpha_k(k^*) = k^* f'(k^*)/f(k^*)$ is the elasticity of output with respect to capital (at steady state).
- Thus, we find:

$$\frac{s}{y^*} \frac{\partial y^*}{\partial s} = \frac{\alpha_k(k^*)}{1 - \alpha_k(k^*)}$$

- If we have information about the elasticity of output with respect to capital, we can gauge the output elasticity with respect to the saving rate.
 - Assume that markets are competitive and capital earns its marginal product:

$$\frac{\partial Y}{\partial K} = ALf'(k) \underbrace{\left[\frac{1}{AL} \right]}_{= \frac{\partial k}{\partial K}} = f'(k)$$

The Effect on Output per Unit of Effective Labor in the Long Run IV

An Example of Using Approximations to Gauge Effects

- Then, the share of capital in output in steady state is $k^* f'(k^*)/f(k^*) = \alpha_k(k^*)$.
- Therefore, in a competitive environment, the capital share of output is a good estimate of $\alpha_k(k^*)$.
- For many countries, the share of capital in output is about 1/3, which implies that α_k is about 1/2.
- This means that an increase in the saving rate by 10%, e. g., from 20% to 22%, increases output per worker in the long run by about 5% relative to the original path.

The Speed of Convergence I

- In addition to gauging the magnitude of changes, we would also like to know how fast the associated changes occur.
- We assess this by considering (Taylor-type) approximations around the steady state, for simplicity for k rather than y :

$$\Delta k_{t+1} = k_{t+1} - k_t = \underbrace{\frac{1}{1+g+n}}_{\simeq 1} [sf(k_t) - (g+n+\delta)k_t]$$

- We also need to remember that in the steady state it holds that $sf(k^*) = (g+n+\delta)k^*$.

The Speed of Convergence II

- Take the derivative of Δk_{t+1} with respect to k_t and evaluate at $k = k^*$:

$$\begin{aligned}
 \frac{\partial \Delta k_{t+1}}{\partial k_t} \Big|_{k=k^*} &= \underbrace{\frac{1}{1+g+n}}_{\simeq 1} [s f'(k_t) - (g+n+\delta)] \\
 &= \underbrace{\frac{1}{1+g+n}}_{\simeq 1} \left[\frac{(g+n+\delta)k^*}{f(k^*)} f'(k_t) - (g+n+\delta) \right] \\
 &= \underbrace{\frac{g+n+\delta}{1+g+n}}_{\simeq 1} [\alpha_k(k^*) - 1]
 \end{aligned}$$

- This implies that (as first-order approximation):

$$\Delta k_{t+1} \simeq -[1 - \alpha_k(k^*)] (g+n+\delta) [k_t - k^*]$$

The Speed of Convergence III

- What is the above for *approximately realistic* numbers?
- Often $g + n + \delta = 0.06$ is used as a rough guess, with both g and n equal to 1–2% and δ around 3–4%.
- With a capital share of $1/3$, this implies that $[1 - \alpha_k(k^*)](g + n + \delta) = 0.04$.
- This means that from one period (year with these numbers) k – and also y – move about 4% of the remaining distance to the steady-state value k^* – or y^* .
- This also implies that it takes about 17 years to get halfway towards the steady-state values (or in per-worker terms: towards the balanced growth path).

The Solow Model and Vast Output-per-Worker Differences I

- The Solow model features two sources of variation – either over time or across countries – in output per worker:
 - Differences in the capital stock per worker $\frac{K}{L}$
 - Differences in productivity A
- The previous considerations have already hinted at the fact that only differences in technology may account for large differences.
- This holds true in particular when capital earns its marginal product.

The Solow Model and Vast Output-per-Worker Differences II

- The direct route of seeing that capital differences cannot be the source:
 - Suppose output per worker $\frac{Y}{L}$ is x times higher in one country than in another.
 - This means log outputs per worker differ by a factor $\ln x$.
 - Given output elasticity α_k , log capital per worker must differ by $\frac{\ln x}{\alpha_k}$.
 - This in turn implies that capital per worker must differ by a factor $e^{\frac{\ln x}{\alpha_k}}$, equal to x^{1/α_k} .
- Now, output per worker differs by a magnitude of $x = 10$ or even more across different parts of the world.
- Explaining these differences – under the assumptions used – with capital-per-worker differences requires differences of the order $10^{\frac{1}{1/3}} = 1000$.
- However, capital-output ratios do not vary that vastly across countries, they differ maybe by a factor 2 or 3, which translates in capital-per-worker differences of about 20 or 30.

The Solow Model and Vast Output-per-Worker Differences III

- The indirect route – pioneered by Lucas (1990):
 - If markets are competitive, the rate of return equals its marginal product $f'(k)$ minus depreciation δ .
 - For the example of the Cobb-Douglas production function, the elasticity of the marginal product of capital with respect to output is $-\frac{1-\alpha}{\alpha}$.
 - For $\alpha = 1/3$, we thus have an elasticity of 2.
 - Thus, in this case a tenfold difference in output per worker arising from differences in capital per worker implies a hundredfold difference in the marginal product of capital.
 - The rates of return on capital are not that different across countries.
- Thus, in the end the big differences can only be explained through differences in technology – technology, however, is considered exogenous in the Solow model: Endogenous Growth Theory.

Growth Accounting I

- This is a convenient point to discuss the Solow model in **continuous time**, denoting with $\dot{X}(t) = \frac{dX(t)}{dt}$:

$$\dot{A}(t) = gA(t), \quad A(0) > 0$$

$$\dot{L}(t) = nL(t), \quad L(0) > 0$$

$$Y(t) = F(K(t), A(t)L(t))$$

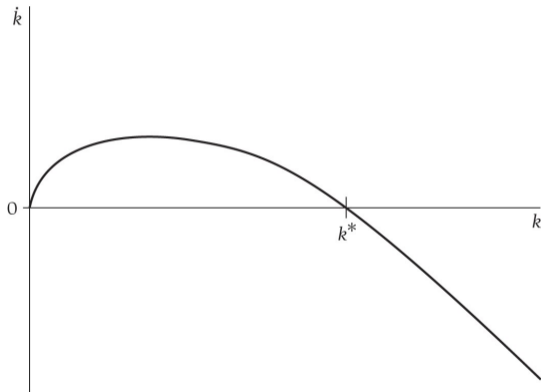
$$\begin{aligned} \dot{K}(t) &= sY(t) - \delta K(t), \quad 0 < s \leq 1 \\ &= [Y(t) - C(t)] - \delta K(t) \end{aligned}$$

Growth Accounting II

- Define $k(t) = \frac{K(t)}{A(t)L(t)}$, and use $\dot{A}(t)/A(t) = g$, $\dot{L}(t)/L(t) = n$ and the chain rule to obtain:

$$\begin{aligned}
 \dot{k}(t) &= \frac{\dot{K}(t)}{A(t)L(t)} - \frac{K(t)}{[A(t)L(t)]^2} [A(t)\dot{L}(t) + \dot{A}(t)L(t)] \\
 &= \frac{\dot{K}(t)}{A(t)L(t)} - \frac{K(t)}{A(t)L(t)} \frac{\dot{A}(t)}{A(t)} - \frac{K(t)}{A(t)L(t)} \frac{\dot{L}(t)}{L(t)} \\
 &= \frac{sY(t) - \delta K(t)}{A(t)L(t)} - k(t)g - k(t)n \\
 &= s \frac{Y(t)}{A(t)L(t)} - (g + n + \delta)k(t) \\
 \dot{k}(t) &= sf(k(t)) - (g + n + \delta)k(t)
 \end{aligned}$$

Growth Accounting III

Figure 5: The phase diagram for k in the Solow model

Growth Accounting IV

- In continuous time it is also “easier” to talk about a Taylor approximation of $\dot{k}(t) = sf(k(t)) - (g + n + \delta)k(t)$ around $k = k^*$:

$$\dot{k}(t) \simeq \underbrace{\left[\frac{\partial \dot{k}}{\partial k} \Big|_{k=k^*} \right]}_{=-\lambda} (k(t) - k^*)$$

- With λ (why is $\lambda > 0$?) as just defined, this implies that:

$$k(t) \simeq k^* + e^{-\lambda t} [k(0) - k^*]$$

- Doing the calculations shows, compare the discrete-time version above, that:

$$\lambda = [1 - \alpha_k(k^*)](g + n + \delta)$$

Growth Accounting V

- But now, to growth accounting:

$$Y(t) = F(K(t), A(t)L(t))$$
$$\dot{Y}(t) = \frac{\partial Y(t)}{\partial K(t)} \dot{K}(t) + \frac{\partial Y(t)}{\partial A(t)} \dot{A}(t) + \frac{\partial Y(t)}{\partial L(t)} \dot{L}(t),$$

with $\frac{\partial Y}{\partial L} = \left[\frac{\partial Y}{\partial AL} \right] A$ and $\frac{\partial Y}{\partial A}$ defined analogously.

- Now divide both sides by $Y(t)$ and rearrange:

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{K(t)}{Y(t)} \frac{\partial Y(t)}{\partial K(t)} \frac{\dot{K}(t)}{K(t)} + \frac{L(t)}{Y(t)} \frac{\partial Y(t)}{\partial L(t)} \frac{\dot{L}(t)}{L(t)} + \frac{A(t)}{Y(t)} \frac{\partial Y(t)}{\partial A(t)} \frac{\dot{A}(t)}{A(t)}$$
$$\equiv \alpha_K(t) \frac{\dot{K}(t)}{K(t)} + \alpha_L(t) \frac{\dot{L}(t)}{L(t)} + R(t)$$

- Here $\alpha_K(t)$ and $\alpha_L(t)$ are the elasticities of output with respect to capital and labor.

Growth Accounting VI

- An important quantity, known as **Solow residual**, is the last term above:

$$R(t) = \frac{A(t)}{Y(t)} \frac{\partial Y(t)}{\partial A(t)} \frac{\dot{A}(t)}{A(t)}$$

- Using $\alpha_K + \alpha_L = 1$ (why?) one can transform the above equation to:

$$\frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{L}(t)}{L(t)} = \alpha_K(t) \left[\frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)} \right] + R(t)$$

- The growth rates of output Y , labor L and (also) capital K are easy to measure.
- If capital earns its marginal product, then we know that α_K can be measured as the income share of capital.
- This allows to back out the Solow residual $R(t)$ – a proxy for the growth contribution of technological progress.
- These considerations are the starting point for many iterations and improvements of growth accounting exercises (different types of capital and labor,...).

Convergence I

- The Solow model has been one starting point of an **enormous empirical literature** on economic convergence: Do poorer countries grow faster? [If you want to do empirical macroeconomic work, this is probably a good entry point.]
- Three reasons why the Solow model leads to this “prediction”:
 - First, the Solow model predicts that countries converge to their BGP: If poorer countries are just further away from “the same” BGP...
 - Second, the return to capital is lower in countries with higher capital per worker: Incentives for capital to flow from rich to poor countries (but remember Lucas, 1990).
 - Third, there might be lags in technology adoption, meaning that poorer countries will catch up once they get access to “latest A”.
- Two seminal (simple empirical papers) studying the issue are **Baumol (1986)** and (as a comment) **DeLong (1988)**.

Convergence II

- Baumol (1986) considers a sample of 16 industrialized countries with data available from 1870–1979 and runs the following regression:

$$\ln \left[\left(\frac{Y}{N} \right)_{i,1979} \right] - \ln \left[\left(\frac{Y}{N} \right)_{i,1870} \right] = \beta_0 + \beta \ln \left[\left(\frac{Y}{N} \right)_{i,1870} \right] + \epsilon_i,$$

where $\ln \left[\left(\frac{Y}{N} \right) \right]$ is real GDP per worker.

- **Unconditional** convergence corresponds to a negative value of β :
 - A coefficient $\beta = -1$ corresponds to $\ln \left[\left(\frac{Y}{N} \right)_{i,1979} \right]$ being uncorrelated to the 1870 value.
 - A coefficient $\beta = 0$ implies that growth is uncorrelated to initial income: No convergence.

Convergence III

- Baumol (1986) obtains the following estimation results:

$$\ln \left[\left(\frac{Y}{N} \right)_{i,1979} \right] - \ln \left[\left(\frac{Y}{N} \right)_{i,1870} \right] = 8.457 - \underset{(0.094)}{0.995} \ln \left[\left(\frac{Y}{N} \right)_{i,1870} \right],$$

$R^2 = 0.87, \quad \text{s.e.e.} = 0.15$

Convergence IV

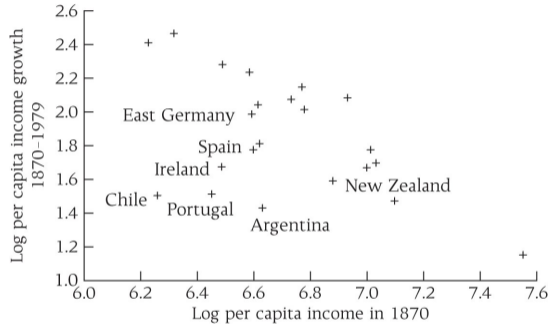
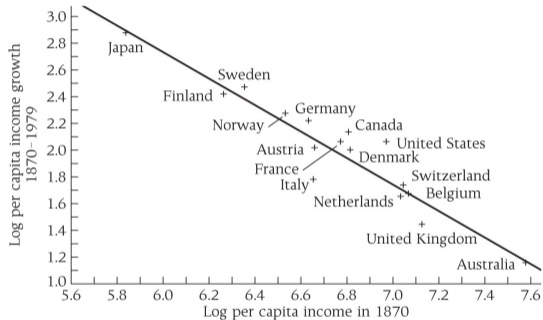


Figure 6: Initial income and subsequent growth: Baumol (1986) left and DeLong (1988) right

Convergence V

- DeLong (1988) offers some comments why maybe Baumol's results are too good to be true:
 - **Sample selection:** The historical data available are mostly for the nowadays most industrialized, rich countries.
 - One way to overcome this **sample selection bias** is to choose the sample not based on the dependent variable; data for all countries are not available for the considered period.
 - DeLong (1988) considers the richest countries in 1870: All countries at least as rich as the second poorest country in 1870 in Baumol (1986), i. e., Brazil.
 - This leads to the inclusion of seven more countries (and dropping Japan).
 - **Measurement error:** Estimates of per-capita income in 1870 are imprecise.
 - Measurement error can tilt results towards convergence: If income in 1870 is overstated, subsequent growth is understated.

Convergence VI

- DeLong (1988) “estimates” a model with errors in variables:

$$\ln \left[\left(\frac{Y}{N} \right)_{i,1979} \right] - \ln \left[\left(\frac{Y}{N} \right)_{i,1870} \right]^* = \beta_0 + \beta \ln \left[\left(\frac{Y}{N} \right)_{i,1870} \right]^* + \epsilon_i$$

$$\ln \left[\left(\frac{Y}{N} \right)_{i,1870} \right] = \ln \left[\left(\frac{Y}{N} \right)_{i,1870} \right]^* + u_i$$

- Of course, $\ln \left[\left(\frac{Y}{N} \right)_{i,1870} \right]^*$ is not observed, so this equation cannot be estimated directly, but DeLong argues that one has a rough idea of the quality of 1870 data:

Convergence VII

- He discusses that knowledge concerning the standard error σ_u of initial income allows estimating the model.
 - $\sigma_u = 0.01$, e. g., means that initial income is measured on average with a 1% error, $\sigma_u = 0.5$ means an average error of 50%.
 - In his sample, the estimate of β tends to 0 for $\sigma_u \simeq 0.15$ and reaches $+1$ – divergence rather than convergence – for $\sigma_u \simeq 0.2$.
 - This means that Baumol's results are not robust.
- Since these early papers an enormous literature finds – despite all the effort – only mixed and partly inconclusive or contradictory evidence for convergence.

Convergence VIII

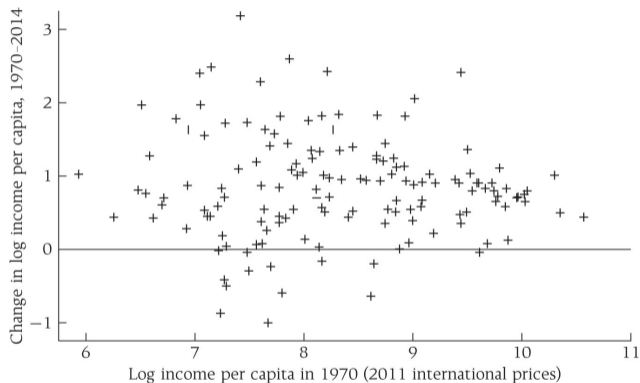


Figure 7: Initial income and subsequent growth in a large sample; world without middle eastern oil producers

Next Steps with the Solow Model

- The Solow model has been extended in many directions...[and continues to be extended]
- The role of human capital: [Mankiw *et al.* \(1992\)](#)
- The implications of natural resources (and/or land)
- Adding emissions as second, negative output: The Green Solow Model of [Brock and Taylor \(2010\)](#)
- Variations of the Solow model continue to be the starting point for many empirical investigations of growth and convergence processes.