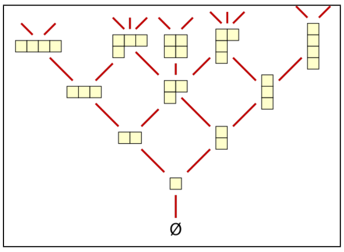
Schur-Weyl Duality: Linking Permutations and Spherical Harmonics

Representation Theory · Harmonic Analysis

A *representation* of a group G (or G*-module*) is a vector space V together with a group homomorphism $\phi \colon G \to \mathrm{GL}(V)$. In other words, a group representation is an explicit way to write elements of a group using invertible matrices (if V is finite dimensional). We can define *submodules* in the natural way: A subspace $W \subseteq V$ is a submodule if $\phi(g)(W) \subseteq W$ for all $g \in G$, i.e., restricting ϕ to the subspace W results in another representation. We call a representation without non-trivial submodules *irreducible*. As with most "nice" algebraic objects, we can show that we can decompose all (finite-dimensional) representations into irreducible submodules

$$V = W_1 \oplus \cdots \oplus W_k.$$



Lattice of partitions

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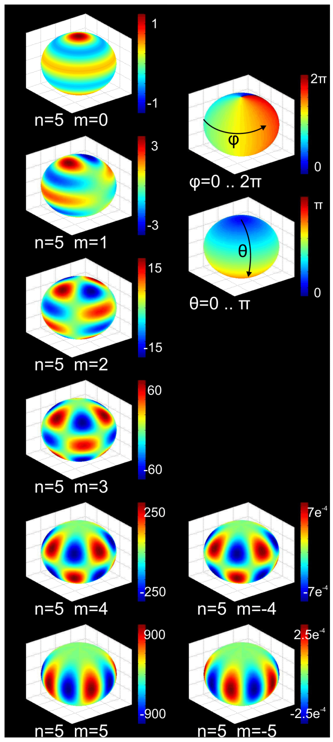

Spherical harmonics

Curiously, any finite group only has finitely many non-isomorphic irreducible representations, and they correspond exactly to the conjugacy classes of G.

The representation theory of permutation groups S_n is very well understood: The conjugacy class of a permutation is determined by its cycle type, i.e., a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{N}^k$ of n. For every partition λ we know exactly how to construct the corresponding irreducible module (called *Specht-module*).

The representation theory of classical Lie groups such as $\mathrm{GL}(n)$ and $\mathrm{SO}(n)$ is deeply linked to the representation theory of S_n, captured by *Schur-Weyl-duality*. Consider the joint representation of S_k and $\mathrm{GL}(n)$ on the tensor space

$$V = (\mathbb{C}^n)^{\otimes k} = \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n,$$

where S_k acts by permuting the factors

$$\sigma(v_1 \otimes \cdots v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)},$$

and $\mathrm{GL}(n)$ by simultaneous matrix multiplication

$$g(v_1 \otimes \cdots \otimes v_k) = gv_1 \otimes \cdots \otimes gv_k.$$

The Schur-Weyl-duality theorem states that V decomposes into tensor products of irreducibles S^λ of S_k and T^λ of $\mathrm{GL}(n)$:

$$(\mathbb{C}^n)^{\otimes k} = \bigoplus_{\lambda} S^\lambda \otimes T^\lambda,$$

where λ sums over partitions of k with at most n parts. I.e., the irreducible modules of permutation groups exactly determine the irreducible modules of general linear groups.

Main tasks

Work out the classical representation theory up and including this theorem, continuing slightly further to the representation theory (and as such harmonic analysis) of $\mathrm{O}(n)$.

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