The Asymptotic Spectrum of Graphs

Graph Theory · Metric Spaces · Optimization

Let $G = (V, E)$ be a graph. The *strong power* $G^{\boxtimes k}$ of G is the k-times *strong product* of G with itself, where $G \boxtimes G$ has vertices $V \times V$ and an edge $\{(a, x), (b, y)\}$ if $(a = b \vee ab \in E)$ and $(x = y \vee xy \in E)$. We denote the *independence number* of G by $\alpha(G)$. The *Shannon Capacity* of G is the limit

$$\Theta(G) \coloneqq \lim_{n\to\infty} \alpha(G^{\boxtimes n})^{1/n}.$$

The Shannon capacity models the maximum zero-error data transmission rate for sending data through an inaccurate channel, and remains one of the least understood graph parameters.

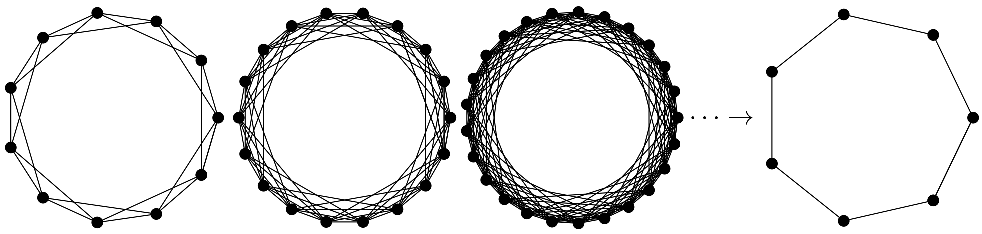
While seemingly unrelated, Jeroen Zuiddam found a connection between Θ and *Strassen duality* in 2019, which was introduced for studying the complexity of matrix multiplication algorithms. In a quick series of follow-ups, the vast majority of methods for investigating Θ became unified in what is now called *asymptotic spectrum duality*. We say that a function F, which assigns every graph a nonnegative real, lies in the asymptotic spectrum $\mathcal{X}$ if

- $F(G \sqcup H) = F(G) + F(H)$ (additivitiy. $\sqcup$ is the disjoint union.)
- $F(G \boxtimes H) = F(G)F(H)$ (multiplicativity.)
- $F(E_n) = n$ (normalization. E_n is the graph with n isolated vertices.)

Some of the (very few) known elements of $\mathcal{X}$ are the Lovàsz Theta function ϑ, the fractional Haemers bound, the projective rank and the fractional clique covering number $\overline{\chi}_f$, as well as interpolations of these. While the Shannon capacity Θ does not lie in $\mathcal{X}$, Zuiddam showed it is exactly the minimum taken over all elements of $\mathcal{X}$:

$$\Theta(G) = \min_{F\in\mathcal{X}} F(G),$$

i.e., every element of the asymptotic spectrum gives an upper bound on Θ. To understand the asymptotic spectrum better, mathematicians are now starting to investigate limits of sequences of graphs converging according to the asymptotic spectrum distance $d(G, H) \coloneqq \sup_{F\in\mathcal{X}} |F(G) - F(H)|$ [de Boer, Buys, Zuiddam 2024]. Curiously, we can find non-trivial sequences of growing graphs which converge to finite graphs:



While this appears to complicate things at first, it can be easier to find large independent sets in the approximating sequence, which we can then "round down" to an independent set in a power of the cycle graph.

Main tasks

Research the (very young) literature, work out the main definitions and and proofs. Potentially implement the algorithm searching for large stable sets in the approximating graph sequences.

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