Factorizations of matrices with non-negative integer entries

Let $(M, \cdot)$ be a commutative monoid and $m \in M$. A *factorization* of m is an expression of m as a product of irreducible elements in M (as usual, an element $c \in M$ is called irreducible if $c = a \cdot b$ implies that a or b is a unit of M).

The aim of this thesis is to study the factorization behavior of the monoid $M = \mathbb{N}_0^{n\times n}$. In their bachelor theses [1, 2] Johannes Schmucker and Markus Tripp were able to lay the groundwork for the 2×2 case. Tripp characterized the zero-divisors, units, and factorizable matrices. Her further showed that the irreducibles split into two main classes which he called (non-)FE-triangular. Two matrices A and B are *factorization equivalent* (short: FE) if there are units U and V such that $A = UBV$ or or $A = UB^tV$. Note that, we can recover all factorizations of B from the factorizations of A and vice versa. For the irreducible 2×2-matrices there is a notable difference depending on whether the matrix has a zero entry or not. In the first case, the matrix is factorization equivalent to a triangular matrix (FE-triangular). The FE-triangular irreducibles are elementary matrices, a result which Tripp shows even for arbitrary n. For non-FE-triangular matrices he shows that all their factors are also non-FE-triangular. For the 2×2 matrices this implies that their entries are strictly positive. Johannes Schmucker studied factorization of matrices with entries in $\mathbb{N}$. Note that $\mathbb{N}^{n\times n}$ is only a semigroup and not a monoid. Nevertheless, the notions of irreducibility and factorizations transfer to this setting. Schmucker develops irreducibility criteria for these matrices in terms of the determinant and the entry constellations. He also provides algorithmic approaches to determine all factorizations of a matrix. Tripp transferred Schmucker's ansatz to deduce irreducibility criteria to the 2×2 matrices over $\mathbb{N}_0$.

The aim of this thesis is to further develop the theory on factorizations of matrices over $\mathbb{N}_0$ (and $\mathbb{N}$). Building on the findings of Schmucker and Tripp, the following problems should be addressed:

- Finding a full characterization of irreducibles in the 2×2 case.
- Can you develop a (reasonably fast) algorithm to determine all factorizations of a matrix
- Examine to what extend the theory transfers to the 3×3 case.
- What can you say about the general $n \times n$ case?

[1] Johannes Schmucker, *Faktorisierung von Matrizen über* $\mathbb{N}_0$, 2023, Bachelorarbeit.

[2] Markus Tripp, *Factorization of square matrices over* $\mathbb{N}_0$, 2023, Bachelor Thesis.

For further details and literature talk to

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