One-dimensional bad noetherian rings

A noetherian, local domain D is sometimes referred to as "bad" if it is

1. one-dimensional (all non-zero prime ideals are maximal)
2. its integral closure D' is **not** finitely generated as D-module, and
3. D is unibranched (D' is local).

Let us have a closer look at the second condition: The integral closure of D is defined as follows. Let S be an overring of D and $a \in S$. An element a of S is called *integral over* D if it satisfies $f(a) = 0$ for some monic polynomial $f \in D[x]$. The *integral closure of D in S* is defined as

$$\mathsf{Cl}_S(D) = \{a \in S \mid a \text{ integral over } D\}.$$

It is known that $\mathsf{Cl}_S(D)$ is a subring of S [2, Satz 3.1.4], in particular, $\mathsf{Cl}_S(D)$ is a ring. If $S = K$ is the quotient field of D, we call $D' = \mathsf{Cl}_K(D)$ the *integral closure* of D.

A *module* over D is an additive group M equipped with an additional scalar multiplication $\cdot\colon D \times M \to M$ such that for all $m, n \in M$ and all $\lambda, \nu \in R$ the following equalities hold: $(\lambda + \nu) \cdot m = \lambda \cdot m + \nu \cdot m$, $\lambda(m + n) = \lambda \cdot m + \lambda \cdot n$, $(\lambda \cdot \nu) \cdot m = \lambda \cdot (\nu \cdot m)$, and $1_D \cdot m = m$. This is a generalization of vector spaces. Modules over fields are vector spaces. A module is *finitely generated* over D if there exist $m_1, \ldots, m_n \in M$ such that every element of M can be written as a D-linear combination of the m_i.

There is a famous construction of Yasno Akizuki [1] from 1935 of a "bad" noetherian ring. The original article is written in german, but uses outdated mathematical language (and there is only a bad quality scan available). Miles Reid provides an english version of this construction [4] which uses modern terminology. The aim of this thesis is to research the literature, work through the construction and explain it in thorough detail.

Other constructions of such rings can also be found in Bruce Olberding's article [3].

[1] Yasuo Akizuki, *Einige Bemerkungen über primäre Integritätsbereiche mit Teilerkettensatz*, Proc. Phys.–Math. Soc. Japan **17** (1935), 327–336.

[2] Franz Halter-Koch, *Algebra und Zahlentheorie*, Lecture notes (in german).

[3] Bruce Olberding, *One-dimensional bad Noetherian domains*, Trans. Amer. Math. Soc. **366** (2014), no. 8, 4067–4095.

[4] Miles Reid, *Akizuki's counterexample*, 1995, unpublished.

For further details and literature talk to

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