Centralized-Equivalent Pairwise Estimation with Asynchronous Communication Constraints for two Robots

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***Abstract*— Collaboratively estimating the state of two robots under communication constraints is challenging regarding computational complexity and statistical optimality. Previous work only achieves practical solutions by either disregarding parts of the measurements or imposing a communication overhead, being non-optimal or not entirely distributed, respectively.**

In this work, we present a *centralized-equivalent* but *distributed* approach for pairwise state estimation where two agents only communicate when they meet. Our approach utilizes elements from wave scattering theory to efficiently and consistently summarize (pre-compute) past estimator information (i.e., state evolution and uncertainty) between encounters of two agents. This summarized information is then used in a joint correction step taking into account all past information of each agent in a statistically correct way.

This novel approach enables us to distribute the pre-computations of both state evolution and uncertainties on the agents and reconstruct the centralized-equivalent system estimate with very few computations once the agents meet again while still applying all measurements from both agents on both estimates upon encounter. We compare our approach on a real-world dataset against a state of the art collaborative state estimation approach.

I. Introduction

For systems with multiple, autonomous agents, pairwise estimation of states between two agents is key for precise and robust localization in challenging environments. Features like sensor sharing (e.g., propagating global information from a GNSS reception on one agent to the other) or instantaneous capturing of a dynamic scene via shared pose information and the resulting variable baseline-stereo setup [1][2] are only some of the benefits that directly result from an accurate pairwise state estimation across two agents. As a specific real-world example, the variable and ad-hoc baseline formation finds application in e.g., landslide or avalanche monitoring, where two aerial agents can form a flexible, sufficiently large baseline on-demand and use their cameras for joint photogrammetric reconstruction of the dynamic events.

Optimal results in terms of consistency and accuracy of the estimated states of the system can only be achieved with centralized fusion of all information of both agents each time a new reading is processed. This is generally computationally not tractable for small mobile systems with limited computing power. Current approaches either approximate inter-agent correlations or assume them to be unknown. Other approaches that maintain correlations between agents, on the other hand, need to communicate when global pose information becomes available to keep the belief equivalent to the centralized fusion.

Our approach for a centralized-equivalent decoupling scheme utilizing elements of the scattering theory efficiently solves estimation problems with pairwise communication constraints. We show that this theory, typically used in physics, can also be used for pairwise state estimation, requiring communication only on meet-ups, i.e., when one agent can sense the other and locate itself relatively to it.

The analogy between waves traveling through media and estimation problems was partially covered in our previous work [3]. However, the focus was only on interpreting measurements as sections of a scattering medium, leading to remarkably fast covariance pre-integration by concatenating those sections into one medium. A vital aspect of this work is to complete the previous theory with reusable mean pre-computations of the state variables, again by using methods of wave scattering and considering estimates to be waves. Information from measurements summarized in this way can be distributed and directly re-applied by estimators. In particular, the ability of scattering theory to *smooth* estimates, i.e. estimate states of the past with more recent measurements, and to change the initial conditions are essential for the presented distributed estimation approach.

We show in this work that both the covariances and the means can be computed with just a few steps using Scattering Theory. All the advantages like changing of initial conditions and concatenating measurements also carry over to the mean computations. The main limitation is that scattering theory was developed for linear systems. Therefore, we present methods to apply it on non-linear systems that paves the way for efficient distributed estimation in a multitude of realistic estimation problems.

Our contributions are:

- Extending previous work on linear systems [4] to cover efficient mean and covariance pre-computations for non-linear systems by the use of scattering theory (IV-C).
- Centralized-equivalent estimates under asynchronous communication constraints for pairwise distributed state estimation on computationally constrained vehicles (V).
- Comparison of the proposed method to a centralized

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implementation using real data (VI).

II. Related Work

Before introducing the related work, the terms centralized and centralized-equivalent are explained. Centralized estimation refers to an estimation approach, where all measurements of each agent are processed in one entity leading to the statistically best possible beliefs. This approach requires constant communication between the central entity and all the agents. If the communication overhead is intractable, the next best estimation scheme is the centralized-equivalent estimation. Compared to the centralized version, the communication is reduced in some way. However, once the agents can communicate, a belief can be computed equivalent to the centralized version, i.e., a centralized-equivalent belief is achieved. In the past decades, different filter-based approaches for collective multiagent localization have been presented. Previous approaches can be roughly classified as (i) centralized-equivalent (e.g., [5], [6]), (ii) approximated (e.g., [7], [8]), (iii) covariance intersection based methods for unknown correlations (e.g., [9], [10]), (iv) optimizing correlations (e.g., [11]), and (v) graph-based methods (e.g., [12]).

The general challenge in all these approaches remains to *decouple* (statistically) the individual agents to relax the communication constraints, while at the same time *maintain* and account for coupling/cross-correlations between agents to achieve statistically optimal and consistent estimates. Current approaches apply different decoupling strategies at the cost of estimator consistency.

In i) [6], Kia et al. proposed a centralized-equivalent decoupled approach based on passing messages with correction terms after joint or global observations to the rest of the agents in a network.

The approximated decoupled filter approaches (ii) are a reasonable choice for real-world applications in terms of scalability regarding the number of involved agents, communication constraints, and accuracy with respect to centralized equivalent approaches, while not being consistent. Luft et al.'s approach presented in [7] requires communication only when agents meet ($\mathcal{O}(1)$) and the maintenance effort for the interdependencies scales with $\mathcal{O}(N)$ for N agents.

At high sensor rates, as it is the case for systems using an IMU as propagation sensor (in aided inertial systems the rate is typically between 100 Hz — 1 kHz), the maintenance effort was identified as a limitation for large swarms. Therefore, Jung and Weiss [8] proposed the use of common correction buffers, allowing the maintenance cost to scale with $\mathcal{O}(1)$ with increasing number of known agents.

In contrast to these desired properties of [7], [8], one major disadvantage remains: directly or indirectly correlated agents that are not participating in the current observation between two other agents do inherit the information of this observation. Meaning that their beliefs experience no correction despite their (theoretical) coupling via cross correlation terms. The reduced computational complexity of these approaches is often favored over the resulting loss in accuracy.

As for the works [9], [10] of iii), unknown correlations are only an issue if the inter-agent correlation terms are not maintained although the agents interacted in the past. Otherwise, inter-agent correlations can be assumed to be zero if they never met. Similarly in iv) [11], the inter-agent correlations are not maintained and must be inferred via optimization and consistency considerations. They can therefore not be completely recovered.

In our previous work, we used Scattering Theory (ST) [13], [14], [4] to perform covariance pre-integration in a single-agent multi-sensor setup [3] and further developed our findings [15] for single agent invariant filtering approaches [16], [17], [18] to enable statistically consistent covariance pre-integration.

Levy et al. [19] also propose a scattering based distributed estimation strategy, but in contrast to our work, they need to the smooth the distributed estimates in an additional operation before they have the centralized-equivalent estimates.

This work achieves centralized-equivalent accuracy and consistency for a pair of agents in contrast to ii), iii) and iv), while still needing to communicate less than i) (we do not need to communicate global information immediately). Our approach takes past *private* observations of the other agent into account upon a *joint* observation, where a private denotes a measurement that only concerns the local state of an agent, and joint denotes a measurement that concerns both agents.

We achieve centralized-equivalence in two steps by employing efficient pre-computations through the use of the scattering theory distributed on each agent as they move and measure independently. We require communication between the two agents to exchange pre-computations only at meet-ups when joint updates are performed (like, e.g., [7], [8]). In doing so, we do not require extensive bookkeeping or information distribution across the entire swarm of agents - two important drawbacks for other centralized-equivalent approaches, e.g., [5], [6].

The rest of the paper is organized as follows: We first develop the necessary tools for a single agent in Sec. IV. The covariance pre-computations derived in the form of *scattering matrices* are discussed in IV-A, as they are needed for the mean pre-computations. In IV-B, we derive *source vectors* used as pre-computation elements for state mean values. The novel extension to non-linear systems is shown in IV-C. Then, in Sec. V, we bring the elements of the single agents together to a pairwise estimation approach for the mean (V-A) and covariance (V-B) computation. The approach is evaluated on a dataset for differential wheel robots in Sec. VI and in Sec. VII we draw the conclusions.

III. Preliminaries and Definitions

In this section, the fundamental definitions for our approach are presented and the notation is defined. A linear system with additive zero-mean white Gaussian noise $\mathbf{n}_u, \mathbf{n}_y$, respective noise covariances $\mathbf{\Sigma}_u$ and $\mathbf{\Sigma}_y$, state $\mathbf{x}$, measurement $\mathbf{y}$ and control input $\mathbf{u}$ in discrete time is given below. System matrices $\mathbf{F}$ and $\mathbf{B}$ describe the linear propagation and $\mathbf{H}$ the linear measurement. Subscript i refers to the time step:

$$\mathbf{x}_{i+1} = \mathbf{F}_i\mathbf{x}_i + \mathbf{B}_i\mathbf{u}_i + \mathbf{n}_{u,i} \qquad \mathbf{n}_u \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_u) \tag{1}$$
$$\mathbf{y}_i = \mathbf{H}_i\mathbf{x}_i + \mathbf{n}_{y,i} \qquad \mathbf{n}_y \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_y) \tag{2}$$

A different non-linear system with additive zero-mean white Gaussian noise is defined below with the measurement function $h()$ and system propagation function $f()$. Using the same noise, state, control input and measurement variable names.

$$\mathbf{x}_{i+1} = f(\mathbf{x}_i, \mathbf{u}_i) + \mathbf{n}_{u,i}, \qquad \mathbf{n}_u \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_u), \tag{3}$$
$$\mathbf{y}_i = h(\mathbf{x}_i) + \mathbf{n}_{y,i}, \qquad \mathbf{n}_y \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_y). \tag{4}$$

The Jacobians of the measurement and the propagation function with the used linearization points (e.g. subscript $\mathbf{x} = \hat{\mathbf{x}}$, $\hat{\mathbf{x}}$ is an estimate) are defined below. These matrices are later used for linearized systems, hence the same name as in the linear case:

$$\mathbf{F}_i = \frac{\partial f(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}}\bigg|_{\mathbf{x}=\hat{\mathbf{x}}_i, u=u_i} \qquad \mathbf{H}_i = \frac{\partial h(\mathbf{x})}{\partial \mathbf{x}}\bigg|_{\mathbf{x}=\hat{\mathbf{x}}_i} \tag{5}$$

State estimates are described by the mean $\hat{\mathbf{x}}_{i|j}$ and the covariance $\mathbf{P}_{i|j}$. Furthermore, i indicates the time of the estimated variable and $\cdot|j$ indicates that all measurements up until time j are considered. Considering a linear or non-linear system with given initial conditions $\{\mathbf{x}_0, \mathbf{P}_0\}$ and all measurements $[\mathbf{y}_0 \ldots \mathbf{y}_N; \mathbf{u}_0 \ldots \mathbf{u}_{N-1}]$ from time 0 to N, there are three different linear least mean squared estimates for the state $\mathbf{x}_i$ at time i: the *filtered* estimate $\hat{\mathbf{x}}_{i|i}$ which is considering $[\mathbf{y}_0 \ldots \mathbf{y}_i; \mathbf{u}_0 \ldots \mathbf{u}_{i-1}]$, the *predicted* estimate $\hat{\mathbf{x}}_{i|i-1}$ which is considering $[\mathbf{y}_0 \ldots \mathbf{y}_{i-1}; \mathbf{u}_0 \ldots \mathbf{u}_{i-1}]$, and the *smoothed* estimate $\hat{\mathbf{x}}_{i|N}$ which is considering all measurements $[\mathbf{y}_0 \ldots \mathbf{y}_N; \mathbf{u}_0 \ldots \mathbf{u}_{N-1}]$. The estimates $\hat{\mathbf{x}}_{i|i-1}$ and $\hat{\mathbf{x}}_{i|i}$ can be computed with a Kalman Filter or Extended Kalman Filter (EKF), that is also providing the innovations $\mathbf{e}_i = \mathbf{y}_i - \mathbf{H}_i\hat{\mathbf{x}}_{i|i-1}$ or $\mathbf{e}_i = \mathbf{y}_i - h(\hat{\mathbf{x}}_{i|i-1})$ and their covariance $\mathbf{R}_{e,i}$ during the filtering process. The innovations are not used in an implementation but are needed for derivations.

IV. Single Agent Pre-Computations with Scattering Theory

The core aspect of this paper is to consider all private observations two agents may have had between their previous and current encounter in a statistically correct fashion. The method should be equivalent to a fully centralized approach, but with reduced compute and communication requirements. The centralized version updates all beliefs of all agents whenever an agent receives an observation. Our approach is to continuously process all observations of the agents separately (distributed) as they move to pre-computation terms, and then exchange them with the other agent when they meet. Using these exchanged pre-computations and then applying the joint measurement is statistically equivalent to a centralized approach. For our EKF setup, the above involves precomputations for the covariance and the mean.

A. Covariance Pre-Computations as Scattering Matrices

A quick recapitulation of the previous work on covariance pre-integration [3] will introduce the basic concepts in scattering theory for covariance pre-computations, which are useful for example, to update with delayed measurements or to perform pairwise estimation.

We show how many measurements are combined into one element $\mathcal{S}$, such that all measurements can be later applied in one step. We are considering a non-linear system as in Eq. 3 and 4. First, we bring propagation and update measurements to the same form as scattering matrices by using their respective generators $\mathbf{M}_{t,i}$ or $\mathbf{M}_{m,i}$ in Eq. (8). $(\mathbf{F}, \mathbf{\Sigma}_u)$ and $(\mathbf{H}, \mathbf{\Sigma}_y)$ are the Jacobian and noise covariance of the state dynamics and measurements, respectively. The subscripts t and m indicate *time propagation* and *measurement*. Next, we can combine generators, i.e. measurements, by using the start product [14] Eq. (7) to a single scattering matrix $\mathcal{S}^0_{i,N}$ in Eq. (9), which is in fact the single agent covariance pre-computation. For a given initial covariance $\mathbf{P}_{i,0}$ at time i, we can compute a covariance $\boldsymbol{\mathcal{P}}_{N,i} = \mathbf{P}_{N|N-1}$ at time N considering all measurements between i and N and as shown in Eq. (10). $\mathcal{S}_{i,N}$ (later used for smoothed estimates) differs from $\mathcal{S}^0_{i,N}$ in that the initial conditions are already applied.

$$\mathcal{S} = \mathcal{S}_1 \star \mathcal{S}_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \star \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{6}$$
$$= \begin{bmatrix} A(I-bC)^{-1}a & B + Ab(I-Cb)^{-1}D \\ c + dC(I-bC)^{-1}a & d(I-Cb)^{-1}D \end{bmatrix} \tag{7}$$
$$\mathbf{M}_{t,i} = \begin{bmatrix} \mathbf{F}_i & \mathbf{\Sigma}_u \\ \mathbf{0} & \mathbf{F}_i^T \end{bmatrix} \quad \mathbf{M}_{m,i} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{H}_i^T\mathbf{\Sigma}_y^{-1}\mathbf{H}_i & \mathbf{I} \end{bmatrix} \tag{8}$$
$$\mathcal{S}^0_{i,N} = \mathbf{M}_{t,i} \star \mathbf{M}_{m,i} \star \mathbf{M}_{t,i+1} \star \mathbf{M}_{m,i+1} \star \ldots \star \mathbf{M}_{t,N-1} \tag{9}$$
$$\mathcal{S}_{i,N} = \begin{bmatrix} \mathbf{I} & \mathbf{P}_{i,0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \star \mathcal{S}^0_{i,N} = \begin{bmatrix} \mathbf{\Phi}_{N,i} & \boldsymbol{\mathcal{P}}_{N,i} \\ -\boldsymbol{\mathcal{O}}_{N,i} & \mathbf{\Phi}^T_{N,i} \end{bmatrix} \tag{10}$$

The next set of equations explaining the entries of $\mathcal{S}_{i,N}$ is not required for implementation but is a necessary part of the theory to understand the mean pre-computations. The first equation Eq. (11) describes the closed loop transfer function that is required mainly to compute cross-covariances from innovations to states. Eq. (12) is the covariance of the estimation error after all measurements are applied. Eq. (13) is called the observability Gramian and is also at the same time the covariance of the *adjoint variable*, which is essential for smoothing and will be introduced in Sec. IV-B.

$$\mathbf{\Phi}_{N,i} = \mathbf{\Phi}_p(N,i) = \mathbf{F}_{p,N-1}\mathbf{F}_{p,N-2}\ldots\mathbf{F}_{p,i} \tag{11}$$
$$\boldsymbol{\mathcal{P}}_{N,i} = \mathbf{P}_{N|N-1} \tag{12}$$
$$\boldsymbol{\mathcal{O}}_{N,i} = \sum_{j=i}^{N-1} \mathbf{\Phi}_p^T(j,i)\mathbf{H}_j^T\mathbf{R}_{e,j}^{-1}\mathbf{H}_j\mathbf{\Phi}_p(j,i) \tag{13}$$
$$\mathbf{\Phi}_p(i,i) = \mathbf{I} \qquad \mathbf{F}_{p,i} = \mathbf{F}_i - \mathbf{K}_{p,i}\mathbf{H}_i \tag{14}$$
$$\mathbf{K}_{p,i} = \mathbf{F}_i\mathbf{P}_i\mathbf{H}_i^T\mathbf{R}_{e,i}^{-1} \qquad \mathbf{R}_{e,i} = \mathbf{H}_i\mathbf{P}_i\mathbf{H}_i^T + \mathbf{\Sigma}_y \tag{15}$$

B. Mean Pre-Computations as Source Vectors

The scattering theory has two integral components: the scattering medium and the waves that travel through the medium. In simple terms, we treated covariances as (parts of) scattering media and derived efficient pre-computations for covariances. Here we will treat *estimate means* as waves and achieve similar mean pre-computations. In particular, $\hat{\mathbf{x}}$ can be considered the forward-in-time wave and the *adjoint variable* $\boldsymbol{\lambda}$ as the backward-in-time wave. The adjoint variable is required in the context of *smoothing* and is described at the end of this subsection.

In parallel to covariance pre-computations as in Eq. (7-9), a similar procedure is done for the means. We are considering a linear system as in Eq. (1-2) and describe the differences for a non-linear system in Sec. IV-C. Note, all variables in this subsection are defined in Sec. III.

First, we bring propagation $\mathbf{u}_i$ (subscript t) and update measurements $\mathbf{y}_i$ (subscript m) to the same form as source vectors by using the Eq. (16). Every measurement defines a scattering section $\{\mathcal{S}, \mathbf{s}\}$, consisting of a scattering matrix and a source vector. To combine source vectors of scattering sections $\{\mathcal{S}_1, \mathbf{s}_1\}$ and $\{\mathcal{S}_2, \mathbf{s}_2\}$ to one source vector $\mathbf{s}$ the dot-sum is used, as defined in Eq. (17). The scattering matrices are defined as in Eq. (6).

$$\mathbf{m}_{t,i} = \begin{bmatrix} \mathbf{B}_i \mathbf{u}_i \\ \mathbf{0} \end{bmatrix} \qquad \mathbf{m}_{m,i} = \begin{bmatrix} \mathbf{0} \\ \mathbf{H}_i^T \boldsymbol{\Sigma}_{y,i}^{-1} \mathbf{y}_i \end{bmatrix} \tag{16}$$

$$\begin{aligned} \mathbf{s} = \mathbf{s}_1 \bullet \mathbf{s}_2 &= \begin{bmatrix} r^+ \\ r^- \end{bmatrix} \bullet \begin{bmatrix} R^+ \\ R^- \end{bmatrix} \\ &= \begin{bmatrix} R^+ \\ r^- \end{bmatrix} + \left(\begin{bmatrix} I & b \\ 0 & d \end{bmatrix} \star \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} \right) * \begin{bmatrix} r^+ \\ R^- \end{bmatrix} \\ &= \begin{bmatrix} R^+ \\ r^- \end{bmatrix} + \begin{bmatrix} A(I - bC)^{-1}(r^+ + bR^-) \\ d(I - Cb)^{-1}(R^- + Cr^+) \end{bmatrix} \end{aligned} \tag{17}$$

Finally, many measurements (i.e., their source vectors) are combined, as in Eq. (18). These are the single agent mean pre-computations for linear systems. After adding the initial conditions, as in Eq. (19), the resulting source vector $\mathbf{s}_{i,N}$ solves two estimation problems simultaneously: The estimation at time N as $\hat{\mathbf{x}}_{N|N-1}$ and the adjoint variable for smoothing at time i as $\boldsymbol{\lambda}_{i|N}$, given all measurements $[\mathbf{y}_i \ldots \mathbf{y}_{N-1}; \mathbf{u}_i \ldots \mathbf{u}_{N-1}]$ and initial conditions $\{\mathbf{x}_{i,0}, \mathbf{P}_{i,0}\}$.

$$\mathbf{s}_{i,N}^0 = \mathbf{m}_{t,i} \bullet \mathbf{m}_{m,i} \bullet \mathbf{m}_{t,i+1} \bullet \mathbf{m}_{m,i+1} \bullet \ldots \bullet \mathbf{m}_{m,N-1} \tag{18}$$

$$\mathbf{s}^b = \begin{bmatrix} \mathbf{x}_{i,0} \\ \mathbf{0} \end{bmatrix} \qquad \mathbf{s}_{i,N} = \mathbf{s}^b \bullet \mathbf{s}_{i,N}^0 = \begin{bmatrix} \hat{\mathbf{x}}_{N|N-1} \\ \boldsymbol{\lambda}_{i|N-1} \end{bmatrix} \tag{19}$$

The innovations approach presented below helps us to understand the meaning of the adjoint variable $\boldsymbol{\lambda}_{i|N-1}$ from Eq. (19) and how smoothed estimates $\hat{\mathbf{x}}_{i|N-1}$ are defined, which will be helpful for the derivations in Sec. V. The covariance is denoted $\langle \cdot, \cdot \rangle$ and $\langle \mathbf{x}_i, \mathbf{e}_j \rangle = \mathbf{P}_{i|i-1} \boldsymbol{\Phi}_p(j,i)^T \mathbf{H}_j^T$ is just presented without derivation.

$$\hat{\mathbf{x}}_{i|N-1} = \sum_{j=0}^{N-1} \langle \mathbf{x}_i, \mathbf{e}_j \rangle \langle \mathbf{e}_j, \mathbf{e}_j \rangle^{-1} \mathbf{e}_j \tag{20}$$

$$\begin{aligned} &= \hat{\mathbf{x}}_{i|i-1} + \sum_{j=i}^{N-1} \langle \mathbf{x}_i, \mathbf{e}_j \rangle \langle \mathbf{e}_j, \mathbf{e}_j \rangle^{-1} \mathbf{e}_j \\ &= \hat{\mathbf{x}}_{i|i-1} + \mathbf{P}_{i|i-1} \sum_{j=i}^{N-1} \boldsymbol{\Phi}_p(j,i)^T \mathbf{H}_j^T \mathbf{R}_{e,j}^{-1} \mathbf{e}_j \\ &= \hat{\mathbf{x}}_{i|i-1} + \mathbf{P}_{i|i-1} \boldsymbol{\lambda}_{i|N-1} \end{aligned} \tag{21}$$

C. Extension to Non-Linear Systems

The Extended Kalman Filter (EKF) is a special case of the linearized Kalman Filter, where the linearization points are taken as the last state estimates. On the same linearized system that the EKF is applied, also the mean computations of the scattering theory can be applied. This only requires certain pre-computations (Eq. (9) and Eq. (18)) to be done, while the EKF is applied to the measurements for the first time. This results in one step re-computations of the EKF means and adjoint variable for new initial conditions (Eq. (26)). In the following derivations, all measurements are processed once with an EKF and therefore all linearization points $\hat{\mathbf{x}}_{i|i}^{\text{lin}}$ and $\hat{\mathbf{x}}_{i|i-1}^{\text{lin}}$ are available. Linearizing is done at a propagated or filtered estimate, $\hat{\mathbf{x}}_i^{\text{lin}} = \hat{\mathbf{x}}_{i|i-1}^{\text{lin}}$ or $\hat{\mathbf{x}}_i^{\text{lin}} = \hat{\mathbf{x}}_{i|i}^{\text{lin}}$, respectively. The linear system for the EKF at the linearization point $\hat{\mathbf{x}}_i^{\text{lin}}$ at time i is described by Eq. (22-23):

$$\begin{aligned} f(\mathbf{x}_i, \mathbf{u}_i) &\approx f\left(\hat{\mathbf{x}}_i^{\text{lin}}, \mathbf{u}_i\right) + \left.\frac{\partial f(\mathbf{x}, \cdot)}{\partial \mathbf{x}}\right|_{\mathbf{x} = \hat{\mathbf{x}}_i^{\text{lin}}} (\mathbf{x}_i - \hat{\mathbf{x}}_i^{\text{lin}}) \\ \mathbf{x}_{i+1} &\approx \hat{\mathbf{x}}_{i+1}^{\text{lin}} + \mathbf{F}_i \Delta \mathbf{x}_i \\ \Delta \mathbf{x}_{i+1} &= \mathbf{x}_{i+1} - \hat{\mathbf{x}}_{i+1}^{\text{lin}} \approx \mathbf{F}_i \Delta \mathbf{x}_i \end{aligned} \tag{22}$$

$$\begin{aligned} h(\mathbf{x}_i, \cdot) &\approx h\left(\hat{\mathbf{x}}_i^{\text{lin}}\right) + \left.\frac{\partial h(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x} = \hat{\mathbf{x}}_i^{\text{lin}}} (\mathbf{x}_i - \hat{\mathbf{x}}_i^{\text{lin}}) \\ \mathbf{y}_i &\approx h\left(\hat{\mathbf{x}}_i^{\text{lin}}\right) + \mathbf{H}_i \Delta \mathbf{x}_i \\ \mathbf{y}_i - h\left(\hat{\mathbf{x}}_i^{\text{lin}}\right) &\approx \mathbf{H}_i \Delta \mathbf{x}_i \end{aligned} \tag{23}$$

An update to the linearization point is then described as:

$$\begin{aligned} \hat{\mathbf{x}}_{i|i}^{\text{lin}} &= \hat{\mathbf{x}}_{i|i-1}^{\text{lin}} + \mathbf{K}_i(\mathbf{y}_i - h(\hat{\mathbf{x}}_{i|i-1}^{\text{lin}})) = \hat{\mathbf{x}}_{i|i-1}^{\text{lin}} + \delta \mathbf{x}_i \\ \mathbf{x}_i - \Delta \mathbf{x}_{i|i} &= \mathbf{x}_i - \Delta \mathbf{x}_{i|i-1} + \delta \mathbf{x}_i \\ \Delta \mathbf{x}_{i|i} &= \Delta \mathbf{x}_{i|i-1} - \delta \mathbf{x}_i \end{aligned} \tag{24}$$

On the linearized system described by Eq. (22-24) the mean computations of the scattering theory, i.e. Eq. (18), can now be applied with slight adaptations to the update measurement source vector $\mathbf{m}_{m,i}$, since we estimate $\Delta \mathbf{x}$ the propagation is canceled out in $\mathbf{m}_{t,i}$:

$$\mathbf{m}_{t,i} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \mathbf{m}_{m,i} = \begin{bmatrix} -\delta \mathbf{x}_i \\ \mathbf{H}_i^T \boldsymbol{\Sigma}_{y,i}^{-1} (\mathbf{y}_i - h(\hat{\mathbf{x}}_i^{\text{lin}})) \end{bmatrix} \tag{25}$$

Given new initial conditions $\{\mathbf{x}_{i,0}^{\text{new}}, \mathbf{P}_{i,0}^{\text{new}}\}$ with $\Delta \mathbf{x}_{i,0} = \mathbf{x}_{i,0}^{\text{new}} - \mathbf{x}_{i,0}$ the smoothed estimates as well as EKF mean estimates can be computed in one step:

$$\mathbf{s}^b = \begin{bmatrix} \Delta \mathbf{x}_{i,0} \\ \mathbf{0} \end{bmatrix} \qquad \mathbf{s}_{i,N} = \mathbf{s}^b \bullet \mathbf{s}_{i,N}^0 = \begin{bmatrix} \Delta \mathbf{x}_N \\ \Delta \boldsymbol{\lambda}_{i|N} \end{bmatrix} \tag{26}$$

$$\hat{\mathbf{x}}_{N|N}^{\text{new}} = \hat{\mathbf{x}}_{N|N}^{\text{lin}} + \Delta \mathbf{x}_N \qquad \text{(for N as update)} \tag{27}$$

$$\hat{\mathbf{x}}_{N|N-1}^{\text{new}} = \hat{\mathbf{x}}_{N|N-1}^{\text{lin}} + \Delta \mathbf{x}_N \qquad \text{(for N as propagation)} \tag{28}$$

$$\hat{\mathbf{x}}_{i|N}^{\text{new}} = \hat{\mathbf{x}}_{i,0}^{\text{new}} + \mathbf{P}_{i,0}^{\text{new}} \Delta \boldsymbol{\lambda}_{i|N} \tag{29}$$

V. Centralized-Equivalent Pairwise State Estimation with Scattering Theory

In the previous section, we described how the two agents generate pre-computations for means and covariances described by Eq.(18) and Eq. (9), respectively. Assume agents A and B are initially correlated at time i and then every agent performs a standard EKF to update its state, assume $\mathbf{x}_A$, with private measurements, say $\mathbf{y}_i^A \dots \mathbf{y}_N^A$, but at the same time also builds up the scattering matrix $\mathcal{S}_A$ and the source vectors $\mathbf{s}_A$ from all its private measurements. Once agent A meets agent B again, they share $\{\mathcal{S}_A, \mathbf{s}_A\}$ and $\{\mathcal{S}_B, \mathbf{s}_B\}$ with each other and update their own state in just two steps with the private measurements of the other agent as changes of their own initial conditions.

A. Centralized-Equivalent Mean Computations

Incorporating all information of agent A to agent B is done by smoothing agent B's state at the initial time i with all of agent A's private measurements to get $\hat{\mathbf{x}}^B_{i|N(A)} = \hat{\mathbf{x}}^B(i, \mathbf{y}_i^A \dots \mathbf{y}_N^A)$ as a first step. Then all of B's own private measurements $\mathbf{y}_i^B \dots \mathbf{y}_N^B$ are applied on top of that changed initial condition as a second step. For better legibility and understanding, we present the following derivations using the regular state notation. For the error-state notation in non-linear systems, the matrices are replaced by their corresponding Jacobians according to Eq. (22-23). The effect of this piecewise linearized representation (linearized at each EKF step) has minimal impact on the performance (c.f. results on real data in Sec. VI), yet allows the use of our proposed scattering theory repertoire for fast (re-)computations. In the following, $\hat{\mathbf{x}}_i = \hat{\mathbf{x}}_{i|i-1}$ and $\tilde{\mathbf{x}}_i = \mathbf{x}_i - \hat{\mathbf{x}}_i$. Deriving $\hat{\mathbf{x}}^B_{i|N(A)}$ and $\mathbf{P}^B_{i|N(A)}$, by applying the innovations $\mathbf{e}$ of A on a joint state vector $\mathbf{z}_i$:

$$\mathbf{z}_i = \begin{bmatrix} \mathbf{x}_i^A \\ \mathbf{x}_i^B \end{bmatrix} \qquad \mathbf{e}_j = \mathbf{H}_{A,j}\tilde{\mathbf{x}}_j^A + \mathbf{n}_{y,j}^A$$

$$\begin{bmatrix} \hat{\mathbf{x}}^A_{i|N(A)} \\ \hat{\mathbf{x}}^B_{i|N(A)} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{x}}_i^A \\ \hat{\mathbf{x}}_i^B \end{bmatrix} + \sum_{j=i}^{N} \langle \mathbf{z}_i, \mathbf{e}_j \rangle \langle \mathbf{e}_j, \mathbf{e}_j \rangle^{-1} \mathbf{e}_j \tag{30}$$

Computing the covariances from states to innovations, and noting that $\hat{\mathbf{x}}_i \perp \tilde{\mathbf{x}}_i$ (so $\langle \hat{\mathbf{x}}_i, \tilde{\mathbf{x}}_i \rangle = 0$) and $\mathbf{n}_{y,j} \perp \{\tilde{\mathbf{x}}_i, \hat{\mathbf{x}}_i\}$ for $j > i$ by definition:

$$\langle \mathbf{z}_i, \mathbf{e}_j \rangle = \langle \begin{bmatrix} \mathbf{x}_i^A \\ \mathbf{x}_i^B \end{bmatrix}, \mathbf{e}_j \rangle \tag{31}$$

$$\begin{aligned} \langle \mathbf{x}_i^A, \mathbf{e}_j \rangle &= \langle \mathbf{x}_i^A, \mathbf{H}_{A,j}\tilde{\mathbf{x}}_j^A + \mathbf{n}_{y,j}^A \rangle && (32) \\ &= \langle \hat{\mathbf{x}}_i^A + \tilde{\mathbf{x}}_i^A, \mathbf{H}_{A,j}\tilde{\mathbf{x}}_j^A + \mathbf{n}_{y,j}^A \rangle \\ &= \langle \hat{\mathbf{x}}_i^A, \mathbf{H}_{A,j}\tilde{\mathbf{x}}_j^A \rangle + \langle \hat{\mathbf{x}}_i^A, \mathbf{n}_{y,j}^A \rangle + \dots \\ &\quad \langle \tilde{\mathbf{x}}_i^A, \mathbf{H}_{A,j}\tilde{\mathbf{x}}_j^A \rangle + \langle \tilde{\mathbf{x}}_i^A, \mathbf{n}_{y,j}^A \rangle \\ &= 0 + 0 + \langle \tilde{\mathbf{x}}_i^A, \mathbf{H}_{A,j}\tilde{\mathbf{x}}_j^A \rangle + 0 \\ &= \langle \tilde{\mathbf{x}}_i^A, \mathbf{H}_{A,j}\mathbf{\Phi}_{p,A}(j,i)\tilde{\mathbf{x}}_i^A \rangle \\ &= \langle \tilde{\mathbf{x}}_i^A, \tilde{\mathbf{x}}_i^A \rangle \mathbf{\Phi}_{p,A}(j,i)^T \mathbf{H}_{A,j}^T \\ &= \mathbf{P}_{A,i}\mathbf{\Phi}_{p,A}(j,i)^T \mathbf{H}_{A,j}^T && (33) \\ \langle \mathbf{x}_i^B, \mathbf{e}_j \rangle &= \langle \tilde{\mathbf{x}}_i^B, \tilde{\mathbf{x}}_i^A \rangle \mathbf{\Phi}_{p,A}(j,i)^T \mathbf{H}_{A,j}^T && (34) \\ &= \mathbf{P}_{BA,i}\mathbf{\Phi}_{p,A}(j,i)^T \mathbf{H}_{A,j}^T && (35) \end{aligned}$$

Now the smoothed estimate is:

$$\begin{aligned} \hat{\mathbf{x}}^B_{i|N(A)} &= \hat{\mathbf{x}}_i^B + \sum_{j=i}^{N} \langle \mathbf{x}_i^B, \mathbf{e}_j \rangle \langle \mathbf{e}_j, \mathbf{e}_j \rangle^{-1} \mathbf{e}_j \\ &= \hat{\mathbf{x}}_i^B + \mathbf{P}_{BA,i} \sum_{j=i}^{N} \mathbf{\Phi}_{p,A}(j,i)^T \mathbf{H}_{A,j}^T \langle \mathbf{e}_j, \mathbf{e}_j \rangle^{-1} \mathbf{e}_j \\ &= \hat{\mathbf{x}}_i^B + \mathbf{P}_{BA,i} \boldsymbol{\lambda}^A_{i|N} && (36) \end{aligned}$$

Given the initial covariance $\mathbf{P}_{BA,i}$ at start time i, it can be seen in Eq. (36) that the adjoint variable of A, which was computed by A as part of $\mathbf{s}_A$, can be directly used to smooth B with A's private measurements. This is a strong contrast to previous distributed approaches, that approximate that private measurements do not change the state of other agents [7], [8]. The corresponding covariance $\mathbf{P}^B_{i|N(A)}$ of the smoothed estimate error is then:

$$\begin{aligned} \tilde{\mathbf{x}}^B_{i|N(A)} &= \mathbf{x}_i^B - \hat{\mathbf{x}}^B_{i|N(A)} \\ &= \mathbf{x}_i^B - (\hat{\mathbf{x}}_i^B + \mathbf{P}_{BA,i}\boldsymbol{\lambda}^A_{i|N}) \\ &= \tilde{\mathbf{x}}_i^B - \mathbf{P}_{BA,i}\boldsymbol{\lambda}^A_{i|N} && (37) \\ \langle \tilde{\mathbf{x}}^B_{i|N(A)}, \tilde{\mathbf{x}}^B_{i|N(A)} \rangle &= \mathbf{P}_{B,i} + \mathbf{P}_{BA,i} \langle \boldsymbol{\lambda}^A_{i|N}, \boldsymbol{\lambda}^A_{i|N} \rangle \mathbf{P}_{BA,i}^T \\ \boldsymbol{\lambda}^A_{i|N} &= \sum_{j=i}^{N} \mathbf{\Phi}_{p,A}(j,i)^T \mathbf{H}_{A,j}^T \langle \mathbf{e}_j, \mathbf{e}_j \rangle^{-1} \mathbf{e}_j && (38) \end{aligned}$$

$$\begin{aligned} \langle \boldsymbol{\lambda}^A_{i|N}, \boldsymbol{\lambda}^A_{i|N} \rangle &= \sum_{j=i}^{N} \mathbf{\Phi}_{p,A}(j,i)^T \mathbf{H}_{A,j}^T \langle \mathbf{e}_j, \mathbf{e}_j \rangle^{-1} \cdot \dots \\ &\quad \langle \mathbf{e}_j, \mathbf{e}_j \rangle \langle \mathbf{e}_j, \mathbf{e}_j \rangle^{-1,T} \mathbf{H}_{A,j} \mathbf{\Phi}_{p,A}(j,i) \\ &= \sum_{j=i}^{N} \mathbf{\Phi}_{p,A}(j,i)^T \mathbf{H}_{A,j}^T \langle \mathbf{e}_j, \mathbf{e}_j \rangle^{-1} \mathbf{H}_{A,j} \mathbf{\Phi}_{p,A}(j,i) \\ &= \boldsymbol{\mathcal{O}}_{A,i|N} && (39) \\ \mathbf{P}^B_{i|N(A)} &= \mathbf{P}_{B,i} + \mathbf{P}_{BA,i} \boldsymbol{\mathcal{O}}_{A,i|N} \mathbf{P}_{BA,i}^T && (40) \end{aligned}$$

$\boldsymbol{\mathcal{O}}_{A,i|N}$ is the observability gramian from the scattering matrix $\mathcal{S}_A$ with already included initial conditions of A that is passed from A to B. For the covariance, state-of-the-art distributed algorithms approximate that there is no change for the passive agents, but the private measurements of the active agent do affect the mean and the covariance c.f. Eq. (36) and Eq. (40). Our approach takes these changes into account with minimal compute and communication requirements.

Now the private measurements of B can be applied on the smoothed initial estimate, leading to a final estimate $\{\hat{\mathbf{x}}^B_{N|N(A,B)}, \mathbf{P}^B_{N|N(A,B)}\}$ at time N that is equivalent to the final estimate of a joint centralized system, although all computations were done in a distributed fashion with a single encounter (i.e., information exchange) of the two agents. Note that $\{\mathcal{S}^{0,B}_{i,N}, \mathbf{s}^{0,B}_{i,N}\}$ were used, which do not include the initial conditions of B, since they are replaced by the smoothed initial estimate. The dots are entries that don't need to be computed and can be omitted.

$$\begin{bmatrix} \mathbf{I} & \mathbf{P}^B_{i|N(A)} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \star \mathcal{S}^{0,B}_{i,N} = \begin{bmatrix} \cdot & \mathbf{P}^B_{N|N(A,B)} \\ \cdot & \cdot \end{bmatrix} \tag{41}$$

$$\begin{bmatrix} \hat{\mathbf{x}}^B_{i|N(A)} \\ \mathbf{0} \end{bmatrix} \bullet \mathbf{s}^{0,B}_{i,N} = \begin{bmatrix} \hat{\mathbf{x}}^B_{N|N(A,B)} \\ \cdot \end{bmatrix} \tag{42}$$

B. Centralized-Equivalent Covariance Computations

The computation of the covariance of the smoothed initial state and the final state of the passive agent B in Eq. (40) and Eq. (41) was derived explicitly to show the contributions of the active agent's private measurements on the passive states. But there is a more direct way, again, using scattering theory, computing the *complete* joint covariance of A and B having processed all private measurements at the final time N.

Two applications show how covariances of smoothing problems are solved with scattering theory. First, the classical smoothing with an extended state is shown, then the smoothing problem for pairwise estimation is solved. For the first application, we want to smooth the state at time k with $k < i$. We extend the state $\mathbf{x}_i$ with the state at time k to get an extended state $\mathbf{z}_{i,k}$ and a new system with initial conditions $\{\hat{\mathbf{z}}_{k,k}, \bar{\boldsymbol{P}}_k\}$ as shown:

$$\mathbf{z}_{i,k} = \begin{bmatrix}\mathbf{x}_i \\ \mathbf{x}_k\end{bmatrix} \qquad \mathbf{z}_{i+1,k} = \boldsymbol{\mathcal{F}}_i \mathbf{z}_{i,k} + \boldsymbol{\mathcal{B}}_i \mathbf{u}_i + \boldsymbol{\mathcal{G}}_i \mathbf{n}_{u,i}$$
$$\mathbf{y}_i = \boldsymbol{\mathcal{H}}_i \mathbf{z}_{i,k} + \mathbf{n}_{y,i}$$
$$\boldsymbol{\mathcal{F}}_i = \begin{bmatrix}\mathbf{F}_i & 0 \\ 0 & I\end{bmatrix} \qquad \boldsymbol{\mathcal{B}}_i = \begin{bmatrix}\mathbf{B}_i \\ 0\end{bmatrix} \quad \boldsymbol{\mathcal{G}}_i = \begin{bmatrix}\mathbf{G}_i \\ 0\end{bmatrix} \quad \boldsymbol{\mathcal{H}}_i = \begin{bmatrix}\mathbf{H}_i & 0\end{bmatrix}$$
$$\hat{\mathbf{z}}_{k,k} = \begin{bmatrix}\hat{\mathbf{x}}_k \\ \hat{\mathbf{x}}_k\end{bmatrix} \qquad \boldsymbol{\mathcal{P}}_{k,k} = \begin{bmatrix}P_k & P_k \\ P_k & P_k\end{bmatrix} = \bar{\boldsymbol{P}}_k$$

Noting that $\hat{\mathbf{x}}_{k|i}$ is the smoothed estimate of $\mathbf{x}_k$, applying all measurements $\mathbf{y}_k \ldots \mathbf{y}_i$ gives the estimates and covariances:

$$\hat{\mathbf{z}}_{i,k} = \begin{bmatrix}\hat{\mathbf{x}}_{i|i} \\ \hat{\mathbf{x}}_{k|i}\end{bmatrix} \qquad \boldsymbol{\mathcal{P}}_{i,k} = \begin{bmatrix}P_{i|i} & P_{ik} \\ P_{ki} & P_{k|i}\end{bmatrix}$$

$\boldsymbol{\mathcal{P}}_{i,k}$ can be computed directly with scattering matrices, because scattering matrices and the joint covariances with interchanged columns satisfy the same Riccati equations [14], [4]. If the columns of $\boldsymbol{\mathcal{P}}_{i,k}$ are interchanged, we get the scattering matrix of the single state system, i.e. $\{\mathbf{x}_i, \mathbf{y}_i, \mathbf{F}_i, \mathbf{B}_i, \mathbf{G}_i, \mathbf{H}_i\}$, with the initial condition of $\bar{\boldsymbol{P}}_k$(i.e., $\bar{\boldsymbol{P}}_k \star \mathcal{S}^0_{k,i}$):

$$\boldsymbol{\mathcal{J}} = \begin{bmatrix}0 & I \\ I & 0\end{bmatrix} \qquad \boldsymbol{\mathcal{P}}_{i,k} = (\bar{\boldsymbol{P}}_k \star \mathcal{S}^0_{k,i})\boldsymbol{\mathcal{J}} = \begin{bmatrix}P_{ik} & P_{i|i} \\ P_{k|i} & P_{ki}\end{bmatrix}\boldsymbol{\mathcal{J}}$$

The second application provides covariances for pairwise estimation as used in this work. Given agents A and B with initial covariance $\bar{\mathbf{P}}_k = [\mathbf{P}_{A,k}, \mathbf{P}_{AB,k}; \mathbf{P}_{BA,k}, \mathbf{P}_{B,k}]$ and generated covariance pre-computations $\mathcal{S}^{A,0}_{k,N}$ and $\mathcal{S}^{B,0}_{k,N}$ by their private measurements, we can compute a centralized-equivalent covariance before the joint update as:

$$\mathcal{S}_A = (\bar{\mathbf{P}}_k \boldsymbol{\mathcal{J}}) \star \mathcal{S}^{A,0}_{k,N} \qquad \boldsymbol{\mathcal{P}}_{k|N(A)} = \mathcal{S}_A \boldsymbol{\mathcal{J}} \tag{43}$$
$$\boldsymbol{\mathcal{P}}_{k|N(A),\mathrm{p}} = \boldsymbol{\mathcal{J}} \boldsymbol{\mathcal{P}}_{k|N(A)} \boldsymbol{\mathcal{J}}$$
$$\mathcal{S}_{B,p} = (\boldsymbol{\mathcal{P}}_{k|N(A),\mathrm{p}} \boldsymbol{\mathcal{J}}) \star \mathcal{S}^{B,0}_{k,N} \qquad \boldsymbol{\mathcal{P}}_{k|N(A,B),\mathrm{p}} = \mathcal{S}_{B,p} \boldsymbol{\mathcal{J}}$$
$$\boldsymbol{\mathcal{P}}_{k|N(A,B)} = \boldsymbol{\mathcal{J}} \boldsymbol{\mathcal{P}}_{k|N(A,B),\mathrm{p}} \boldsymbol{\mathcal{J}} \tag{44}$$

First, we get the covariance of the joint system $\boldsymbol{\mathcal{P}}_{k|N(A)}$ considering all private measurements of A as in Eq. (43), then we also apply all private measurements of B to get $\boldsymbol{\mathcal{P}}_{k|N(A,B)}$ as in Eq. (44). Permutations, i.e. $\boldsymbol{\mathcal{P}}_{\ldots,p}$, for the input covariances are necessary, because agent A and B change the role of active and passive, and then the order in the joint system is also interchanged. As a last step, the joint measurement can now be applied by both agents as a standard EKF update on the joint system with centralized-equivalent covariance $\boldsymbol{\mathcal{P}}_{k|N(A,B)}$.

VI. Results

We have used the UTIAS Multi-Robot Cooperative Localization and Mapping Dataset [20] to evaluate our approach on real data. Differential drive robots move indoors, logging odometry data and range-bearing measurements of known landmarks and other agents when they meet. We used robots 1 and 2 from the first dataset with their trajectories shown in Fig. 1 and Fig. 2. The sampling rate of the odometry was 25 Hz, and the trajectory duration was 375 sec.

In the distributed case, we have one estimator for each robot (A and B), estimating $\mathbf{x} = (\theta, x_r, y_r)$ with the heading and the 2D position, only processing private measurements. The robots do not communicate unless they meet. From wheel encoders, the control input is given as turn rate and velocity $\mathbf{u} = (\omega, v)$. The propagation $f()$ is done by wheel odometry. Known 2D landmarks (x_l, y_l) are measured as $\mathbf{y} = (d, \alpha) = h(x_l, y_l, x_r, y_r)$ with range d and bearing α. Upon meeting, i.e. when one robot can sense the other, a joint measurement becomes available $\mathbf{y} = h(\mathbf{x}^A, \mathbf{x}^B)$ as a range-bearing measurement involving both robot states. In the centralized case, all measurements of both agents are communicated and processed simultaneously in an EKF with a combined robot state as $\mathbf{x} = (\mathbf{x}^A, \mathbf{x}^B)$.

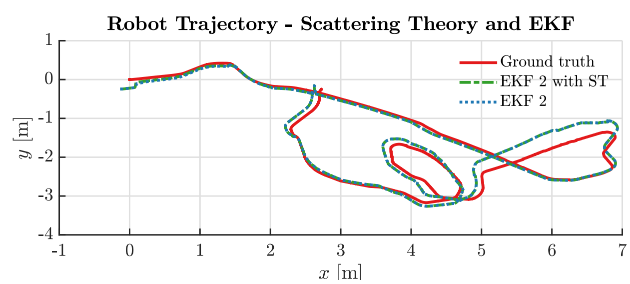

Fig. 1. The trajectory of the first robot is shown in red, the slow EKF computations are shown in blue (EKF 2) and the proposed fast EKF computations are shown in green (EKF 2 with ST). When changing the initial conditions and recomputing the EKF estimates, we achieve the same estimates (i.e., path overlap), indicating that the proposed method can replace computationally intensive re-computations while leading to the same results.

A. Centralized-Equivalent Pairwise Estimation with Ground Robots and Range-Bearing Measurements

We use the proposed methods to perform centralized-equivalent estimation for two robots with asynchronous pairwise communication constraints and compare the results to a fully centralized implementation. Robot one and robot two from the first dataset estimate their state with odometry and known landmark measurements while building up scattering matrices and source vectors. When they meet, they exchange these pre-computed elements and can reproduce a centralized-equivalent joint system update, as if they were connected and exchanging information during the whole time. Fig. 3 shows the estimation error against ground truth, while Fig. 4 shows the error between the estimation methods. There is an order of magnitude lower difference between the two approaches compared to the error of the estimations with respect to ground truth. However, our approach only needs sporadic communication between the robots compared to the fully centralized EKF implementation. There is a maximal error of 2.25 cm and for the heading 0.7 degrees between the two methods (due to space limitations the heading error plot is not depicted).

And finally, in Fig. 5, we compare our approach against Luft et al. [7] in terms of the joint system belief. Their method has the same communication constraints but different distributed covariance pre-computations than our approach. They make certain approximations, and the resulting joint belief is therefore not centralized-equivalent anymore during joint updates. The employed Kullback-Leibler (KL) divergence quantifies the difference between two probability distributions, and should therefore be close to zero if the beliefs are identical. The KL divergence between the joint system and our proposed method (shown in light blue) is close to zero overall, while the values are an order of magnitude higher for the method of Luft et al. [7] (shown in light red). This indicates that the proposed method provides indeed centralized-equivalent beliefs.

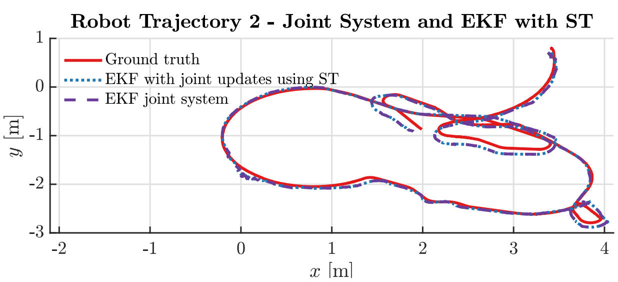

Fig. 2. The trajectory of the second robot (red), the joint estimates (purple) and the proposed centralized-equivalent approach (blue). The estimation behavior is the same, although our approach is restricted in communication, indicating that the proposed method can replace computationally intensive re-computations while leading to the same results.

B. Computation Times

The presented computation times correspond to the same experiment as in the previous section. To describe our approach's computational efficiency, we need to compare it to the computations required to process all measurements of both agents in a joint system at the moment they meet again, which is done in Fig. 6. The first plot shows the computation times for propagation of one agent (0.15 ms) in red and the overhead in each propagation step (0.1 ms) to build the scattering matrices for our approach in blue. The second plot shows how fast our approach computes the joint covariances for joint updates. The longer the agents did not see each other, the more measurements are processed, and therefore more processing time is necessary (maximum of 0.069 s at $t = 166$). On the other hand, if the measurements would be all processed by a joint system only once the agents meet *and not while they are moving*, then the computation takes longer, as shown in the third plot in red (maximum of 1.0 s at $t = 166$). Note that the agents can not communicate until they meet, i.e., can not process the other agent's measurements while moving. The relative computation time of our approach compared to the joint system computation is shown in the last plot, especially when the agents do not meet for long times our approach becomes more efficient (6.8% at $t = 166$ compared to the joint system).

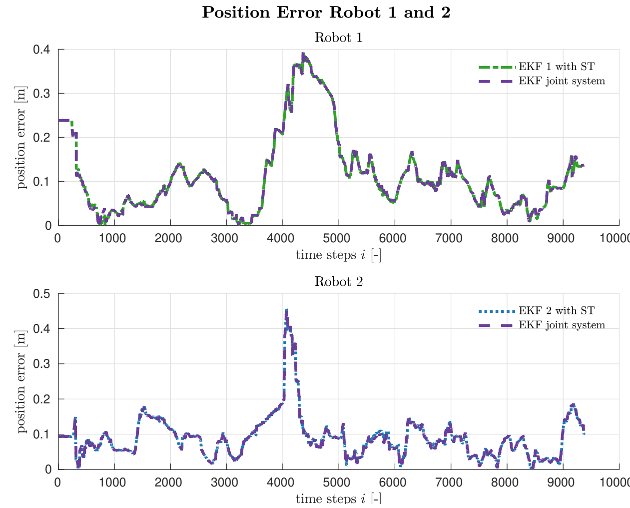

Fig. 3. The estimation error for the position of the joint system and the presented approach is shown. The joint system (purple) and the estimates of the agents (green and blue for robot 1 and 2, respectively). The estimation errors are very close for both approaches, showing that the presented work can achieve the same estimation performance, while performing computations efficiently and only when the agents meet.

VII. Conclusions

We presented a distributed but centralized-equivalent state estimation approach for two robots that have asynchronous pairwise communications constraints. The approach is based on the scattering theory and an analogy to waves traveling through media was presented. In this analogy we first derived the necessary and novel methods for distributed mean pre-computations on non-linear systems and then applied it to pairwise estimation. The combination of many measurements and the ability to change initial conditions in one step enabled us to smooth the state of agents with the measurements of other agents only when they meet, not requiring any constant communication channel to be open yet being able to reconstruct all statistical information from observations the other agent had since the previous meeting. Our novelty is that we extended the previous work on Collaborative State Estimation with constrained and pairwise communication to be statistically truly centralized-equivalent for two robots. Furthermore, we showed that the benefits of pairwise updates are maintained while requiring only very few computations, because measurements can be readily applied with the help of scattering matrices and source vectors. We evaluated our algorithm on real data and showed that the difference to the estimation in a centralized system is an order of magnitude

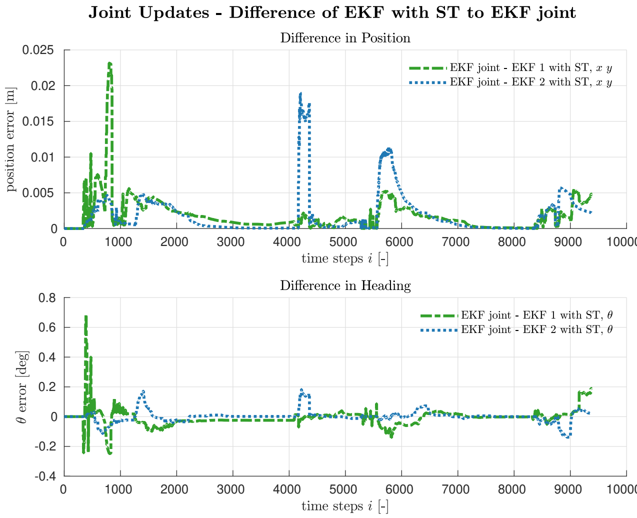

Fig. 4. The plots show the difference in the estimates of the centralized estimator and the two agents (agent 1 in green, agent 2 in blue) performing joint updates with the presented centralized-equivalent approach. In the top plot the difference in x and y is computed as a norm, and in the bottom plot the difference in heading is shown. The error introduced by the presented approach is about an order of magnitude lower than the estimation error itself, comparing the spike at 2.25 cm against a maximum error of 40 cm in Fig. 3.

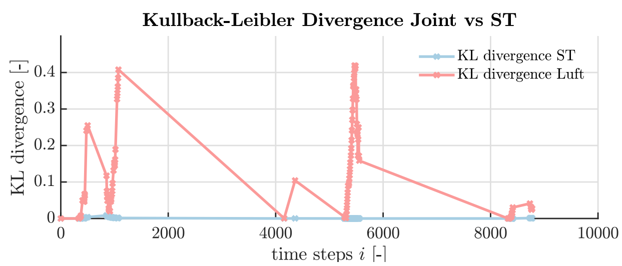

Fig. 5. The symmetric Kullback-Leibler Divergence (KL) quantifies the difference of two beliefs, in our case considering normal distributions. We plotted the KL divergence between the belief of the joint system and the belief of the two agents when they meet and perform a joint update using scattering theory in light blue. As a comparison the KL divergence of an approximating approach of Luft et al. [7], which has the same communication constraints, is also shown in light red. While the difference for our approach is overall very low, the approximations of Luft et al. lead to an order of magnitude higher values, indicating that the proposed method is indeed centralized-equivalent.

smaller than the actual estimation error of both systems.

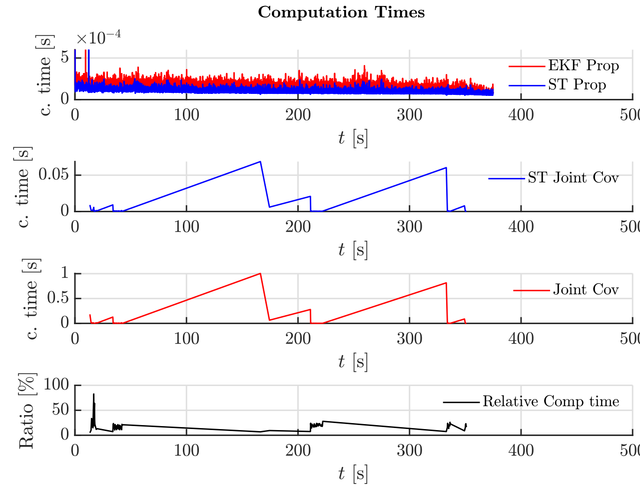

Fig. 6. We show the computational efficiency of our approach compared to a joint system computation. Agents can only exchange measurements when they meet and therefore all measurements would need to processed either with our faster approach (see second plot) or by forming a joint system and reprocessing all measurements on meetup with a joint system (see third plot). Our approach induces an overhead (see blue line in first plot), but reduces the overall computation time on meetup drastically (see relative comparison last plot).

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