A Universally Efficient Dynamic Auction for All Unimodular Demand Types

Satoru Fujishige and Zaifu Yang

Kyoto University (Japan) & University of York (UK)

CED Padova, 10 June 2022

The Main Goal

Consider an auction market where all kinds of indivisible objects/items can be sold. They can be heterogeneous, substitutes, complements, or any mixture of substitutes and complements.

Bidders may demand any number of items. Each bidder has his private valuation over every bundle of objects and may act strategically. The seller has her reserve price for every bundle of items.

How to design an auction mechanism with simple, practical and transparent rules that will allocate objects efficiently, induce bidders to bid truthfully and require as little information from bidders as possible to protect their privacy?

Highlights of Our Contributions

A novel and universal dynamic design is developed and applies to all unimodular demand types of Baldwin and Klemperer (2019) which are a necessary and sufficient condition for the existence of competitive equilibrium and accommodate all kinds of objects.

The auction always induces bidders to bid truthfully and yields an efficient outcome. Bidding sincerely is an ex post perfect Nash equilibrium.

The auction is privacy-preserving and independent of any probability distribution assumption.

The trading rules are simple, practical, transparent and detail-free.

Motivation

The huge volume of the sale of spectrum licenses in the world since early 1990s; Klemperer (2004) and Milgrom (2004). Airport take-off and landing slots, cloud computing band and time allocation, networks, mining rights, treasury bills, key words, pollution permits, etc. Business via auction involves billions and billions of dollars.

Substitutability and complementarity are fundamental properties of goods and services and are pervasive.

Baldwin and Klemperer (2019) found surprisingly that contrary to popular belief, equilibrium is guaranteed for more classes of complements than of substitutes.

Milgrom (2017) says: "Markets for complements can be much harder than markets for substitutes and can require greater planning and coordination." See also Milgrom (2000), Jehiel and Moldovanu (2003), Klemperer (2004), Maskin (2005).

Some Literature

- (Single-Item Auctions) Vickrey (1961), Myserson (1981), Riley and Samuelson (1981), Milgrom and Weber (1982), etc.

- (Multi-Item Auctions for Substitutes) Crawford and Knoer (1981), Kelso and Crawford (1982), Demange et al. (1986), Gul and Stacchetti (2000), Milgrom (2000), Ausubel (2004, 2006), Perry and Reny (2005), and Milgrom and Strulovici (2009).

- (Multi-Item Auctions for Substitutes and/or Complements) Sun and Yang (2009, 2014)

- (Multi-Item Package Auctions) Ausubel and Milgrom (2002) using discriminatory and nonlinear pricing rules

The Model

- $N = \{1, 2, \cdots, n\}$: the set of indivisible items for sale.
  Each item $j \in N$ can be also represented by the j-th unit vector $e(j) \in \mathbb{Z}^N$, where $\mathbb{Z}$ is the set of integers.

- B: a group of m potential bidders.

- Every bidder $j \in B$ has a private valuation function $u^j : \{0, 1\}^N \mapsto \mathbb{Z}_+$ with $u^j(\emptyset) = 0$.

- The seller has a weakly increasing reserve function $u^0 : \{0, 1\}^N \mapsto \mathbb{Z}_+$ with $u^0(0) = 0$. She will not sell any bundle, if the total price for the bundle is less than her reserve value.

- Let $B_0 = B \cup \{0\}$ denote the set of the seller and all bidders.

- Let $\mathcal{M} = (u^j, j \in B_0, N)$ or simply $\mathcal{M}$ represent the market.

- We view a set $S \subset N$ and the corresponding vector $\sum_{h \in S} e(h)$ as the same bundle.
- A price vector $p = (p_1, \cdots, p_n) \in \mathbb{R}^N$ specifies a price p_h for each item $h \in N$. This is a linear and anonymous pricing rule.
- Every bidder $j \in B$ tries to maximize his profit and his demand correspondence $D^j(p)$ is given by

$$D^j(p) = \arg \max_{x \in \{0,1\}^N} \{u^j(x) - p \cdot x\}.$$

- At prices $\mathbb{R}^N$, the seller chooses bundles to maximize her revenues and her demand correspondence $D^0(p)$ is given by

$$\begin{aligned} D^0(p) &= \arg\max_{x \in \{0,1\}^N} \{u^0(x) + p \cdot (\textstyle\sum_{h \in N} e(h) - x)\} \\ &= \arg\max_{x \in \{0,1\}^N} \{u^0(x) - p \cdot x\} \end{aligned}$$

The set $D^0(p)$ contains those bundles that the seller wishes to keep in hand and give her the highest revenues.

- An *allocation* of items in N is a *redistribution* $X = (x^j, j \in B_0)$ of items among all market participants in B_0 such that $\sum_{j \in B_0} x^j = \sum_{h \in N} e(h)$ and $x^j \in \{0,1\}^N$ for all $j \in B_0$.

- At allocation X, agent j receives bundle x^j.

- An allocation $X = (x^j, j \in B_0)$ is *efficient* if $\sum_{j \in B_0} u^j(x^j) \geq \sum_{j \in B_0} u^j(y^j)$ for every allocation $Y = (y^j, j \in B_0)$.

- Given an efficient allocation X, let $R(N) = \sum_{j \in B_0} u^j(x^j)$. We call $R(N)$ *the market value* of the items.

- A competitive or Walrasian equilibrium (p, X) consists of a price vector $p \in \mathbb{R}^N$ and an allocation X such that $x^j \in D^j(p)$ for every $j \in B_0$.

The following two conditions will be imposed on the auction market $\mathcal{M}$:

(A1) *Integer private values*: Every agent $j \in B_0$ knows his/her own utility function $u^j : \{0,1\}^N \to \mathbb{Z}_+$ privately. Bidders can act strategically.

(A2) *Common Unimodular Demand Type*: All agents $j \in B_0$ have the same unimodular demand type $\mathcal{D}$ for their utility functions u^j. This information is public and thus known to the auctioneer.

Demand Type: Locus of Indifference Prices

- W.R.T. any given utility function $u : \{0,1\}^N \to \mathbb{R}$, let the demand set at prices p be given by $D_u(p) = \arg\max_{x\{0,1\}^N}\{u(x) - p \cdot x\}$.
- Following Baldwin and Klemperer (2019), we call the set $\mathcal{T}_u = \{p \in \mathbb{R}^N \mid \sharp D_u(p) > 1\}$ *the locus of indifference prices (LIP)* of the demand set D_u, where $\sharp D_u(p)$ denotes the number of elements in $D_u(p)$.
- This set $\mathcal{T}_u$ concerns those price vectors p at which there are at least two optimal bundles for any agent who has the utility function u.
- A set $S \subseteq \mathbb{R}^N$ is a *polyhedron* if $S = \{x \in \mathbb{R}^N \mid Ax \leq b\}$ for some $m \times n$ matrix A and an m-vector. A polyhedron is a *polytope* if it is bounded.

Demand Type

- A facet of a polyhedron of dimension n is a face that has dimension $n-1$.
- A *facet* of $\mathcal{T}_u$ is an $(n-1)$-dimensional subset F of the set $\mathcal{T}_u$ such that there exist $x, y \in D_u(p)$ with $x \neq y$ for some $p \in F$.
- Intuitively, the set $\mathcal{T}_u$ contains the only prices at which demand can change in response to a price change, and is the union of $(n-1)$-dimensional facets. These facets separate the unique demand regions, in each of which some bundle is the unique demand.
- The *normal vector* to a facet F is a vector which is perpendicular to F at a point in its relative interior.
- A non-zero integer vector is *primitive* if the greatest common divisor of its coordinates is one.

A finite set $\mathcal{D}$ of nonzero primitive integer vectors in $\mathbb{Z}^N$ is a *demand type* $\mathcal{D}$ if $v \in \mathcal{D}$ implies $-v \in \mathcal{D}$ and every facet of the LIP $\mathcal{T}_u$ has normal vectors in the set $\mathcal{D}$ and every element in $\mathcal{D}$ is normal to some facet of the LIP.

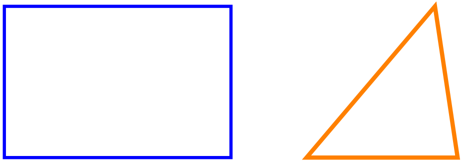

Figure 1: Red and orange lines are facets of the rectangle and triangle, resp.

Example 1: Two bidders and two substitutable items. Valuations:

Bundle	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
*Bidder*1	0	3	4	5
*Bidder*2	0	5	2	6

Both bidders have the same unimodular demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$. See the figure in the next page.

Example 2: Three bidders and two complementary items $A = (1,0)$ and $B = (0,1)$. Valuations:

Bundle	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
*Bidder*1	0	2	2	5
*Bidder*2	0	2	2	5
*Bidder*3	0	1	1	4
Seller	0	1	1	3

All agents have the same unimodular demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(1,1)\}$.

Demand Type: Substitutes

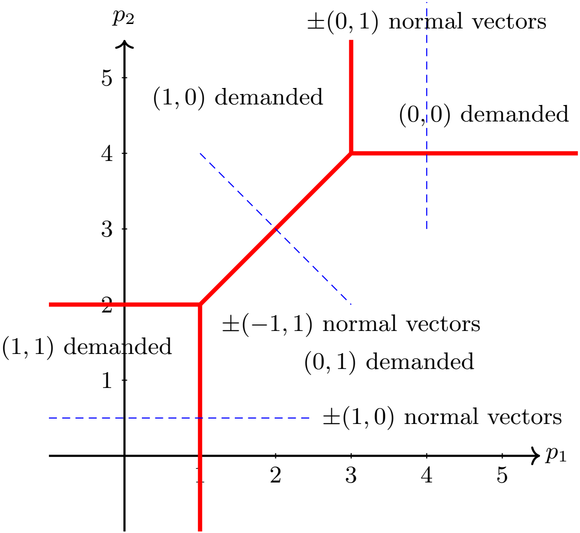

Figure 2: $u(0,0) = 0$, $u(1,0) = 3$, $u(0,1) = 4$, and $u(1,1) = 5$. The red lines denote LIP.

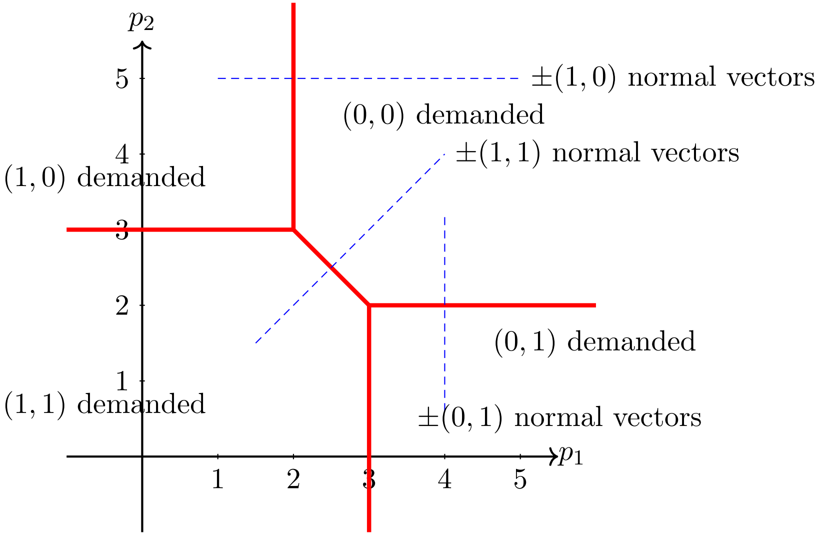

Figure 3: $u(0,0) = 0$, $u(1,0) = u(0,1) = 2$, and $u(1,1) = 5$. The red lines denote LIP.

- Bidders may have different utility functions but can still have the same demand type $\mathcal{D}$ if the physical property of the items is the same to everyone. For instance, chairs are viewed as chairs by all bidders, even though they may valuate chairs differently.
- A square matrix is *unimodular* if all its elements are integral and its determinant is $+1$ or -1. A set of n integer vectors in $\mathbb{R}^N$ is *a unimodular basis* for $\mathbb{R}^N$ if the $n \times n$ matrix which has the n integer vectors as its columns is unimodular.
- Given a utility function u, the associated demand type $\mathcal{D}$ is *unimodular* if every linearly independent subset of $\mathcal{D}$ can be extended to a unimodular basis for $\mathbb{R}^N$.

Baldwin and Klemperer (2019) have identified a variety of demand types and shown the richness of complements.

A demand type $\mathcal{D}$ is *Gross Substitutes* if every vector $x \in \mathcal{D}$ has at most one 1 and at most one -1 and no other nonzero entries. Kelso and Crawford (1982).

Let (S_1, S_2) be a partition of the set N. A demand type $\mathcal{D}$ is *Gross Substitutes and Complements* (GSC) if every vector $x \in \mathcal{D}$ has at most two nonzero components and no other nonzero entries so that if two nonzero components of x have the same sign, then one nonzero component must be indexed by an element in S_1 and the other must be indexed by an element in S_2. Sun and Yang (2006).

A demand type $\mathcal{D}$ is *complements demand type* if every vector $x \in \mathcal{D}$ implies either $x \in \{0, 1\}^N$ or $x \in \{0, -1\}^N$.

- For any set $T \subseteq \mathbb{R}^N$, $\mathrm{Conv}(T)$ denotes its convex hull. For any given $x \in \mathbb{R}^N$, define the integral neighborhood of x as

$$\mathbf{N}(x) = \{y \in \mathbb{Z}^N \mid |y_j - x_j| < 1 \ \text{ for all } \ j = 1, 2, \cdots, n\}$$

- A set $D \subseteq \mathbb{R}^n$ is *integrally convex* if $D = \mathrm{Conv}(D)$ and $x \in D$ implies $x \in \mathrm{Conv}(D \cap \mathbf{N}(x))$, i.e., every point $x \in D$ can be represented as a convex combination of integral points in $\mathbf{N}(x) \cap D$.

- A set $S \subseteq \mathbb{R}^N$ is a *polyhedron* if $S = \{x \in \mathbb{R}^N \mid Ax \leq b\}$ for some $m \times n$ matrix A and an m-vector.

- A polyhedron $S \subseteq \mathbb{R}^N$ is *integral* if all its vertices are integral.

- A function f with a polyhedron domain in $\mathbb{R}^N$ is called *a polyhedral convex function* if it is given as

$$f(x) = \max\{B_j \cdot x + c_j \mid j = 1, \cdots, m\}$$

Integrally Convex Sets

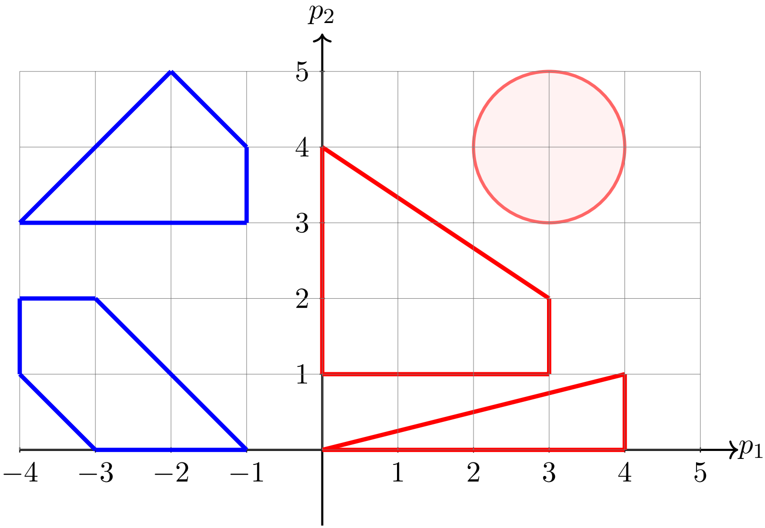

Figure 4: Integrally convex sets and non-integrally convex sets.

- For every $j \in B_0$, define her indirect utility function by

$$V^j(p) = \max_{x \in \{0,1\}^N} \{u^j(x) - p \cdot x\}.$$

- Define the Lyapunov function $\mathcal{L} : \mathbb{R}^N \to \mathbb{R}$ by

$$\mathcal{L}(p) = \sum_{h \in N} p_h + \sum_{j \in B_0} V^j(p).$$

- A function $f : \mathbb{Z}^N \mapsto \mathbb{R}$ is *discrete concave* if for any $\lambda_j \geq 0$, $j = 1, \cdots, t$ and any $x^j \in \mathbb{Z}^N$ for $j = 1, \cdots, t$ with $\sum_{j=1}^t \lambda_j = 1$ and $\sum_{j=1}^t \lambda_j x^j \in \mathbb{Z}^N$ we have

$$f(\lambda_1 x^1 + \lambda_2 x^2 + \cdots + \lambda_t x^t) \geq \sum_{j=1} \lambda_j f(x^j).$$

Lemma 1: For the market model, the Lyapunov function $\mathcal{L}$ is a polyhedral convex function bounded from below.

Theorem 1: Assume that the market model satisfies Assumptions (A1) and (A2). Then the set of competitive equilibrium price vectors forms a nonempty integrally convex polytope.

A search set for a demand type $\mathcal{D}$ is the collection of the zero vector and all nonzero primitive integer vectors $\delta \in \mathbb{Z}^N$ such that we have $\delta \cdot d_i = 0$ for some $n-1$ linearly independent vectors $d_1, \cdots, d_{n-1} \in \mathcal{D}$. The search set is denoted by $\mathcal{SD}$.

Example 1: Two bidders and two substitutable items. Valuations:

Bundle	(0,0)	(1,0)	(0,1)	(1,1)
*Bidder*1	0	3	4	5
*Bidder*2	0	5	2	6

Both bidders have the same unimodular demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$ whose search set equals $\mathcal{SD} = \{(0,0), \pm(1,0), \pm(0,1), \pm(1,1)\}$.

Example 2: Three bidders and two complementary items $A = (1,0)$ and $B = (0,1)$. Valuations:

Bundle	(0,0)	(1,0)	(0,1)	(1,1)
*Bidder*1	0	2	2	5
*Bidder*2	0	2	2	5
*Bidder*3	0	1	1	4
Seller	0	1	1	3

All agents have the same unimodular demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(1,1)\}$ whose search set equals $\mathcal{SD} = \{(0,0), \pm(1,0), \pm(0,1), \pm(1,-1)\}$.

The search set is a family of the zero vector and all nonzero primitive integer vectors $\delta \in \mathbb{Z}^N$ such that δ is a normal vector of a facet of a full-dimensional demand set at some price vector p. Roughly speaking, if we adjust the prices of goods along the direction of an element from the search set, it will not cause dramatic change in demands on goods.

The auction is to search for a minimizer of the Lyapunov function $\mathcal{L}$. Roughly speaking, given an integer price vector $p(t) \in \mathbb{Z}^n$ at time $t \in \mathbb{Z}_+$, the auctioneer asks every bidder i to report his demand $D^i(p(t))$. Then she uses every bidder's reported demand $D^i(p(t))$ to search for a price adjustment δ in the search set so as to reduce the value of the Lyapunov function $\mathcal{L}(p(t) + \delta)$ as much as possible, in the hope of finding the minimum of the Lyapunov function.

Lemma 3: Under Assumptions (A1) and (A2) we have

$$\max_{\delta \in \mathrm{Conv}(\mathcal{SD})} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t) + \delta)\} = \max_{\delta \in \mathcal{SD}} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t) + \delta)\}. \quad (1)$$

Lemma 4: Under Assumptions (A1) and (A2), the set of solutions to the left-side problem of (1) is a nonempty integral polytope.

Lemma 5: Under Assumptions (A1) and (A2), $p^* \in \mathbb{Z}^N$ is a competitive equilibrium price vector if and only if $\mathcal{L}(p^*) \leq \mathcal{L}(p^* + \delta)$ for all $\delta \in \mathcal{SD}$.

The Basic Dynamic Auction: How to Adjust Prices

From the above lemmas, we have the following important relation:

$$\max_{\delta \in \mathrm{Conv}(\mathcal{SD})} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta)\} = \max_{\delta \in \mathcal{SD}} \{\sum_{j \in B_0} \min_{x^j \in D^j(p(t))} x^j \cdot \delta - \sum_{h \in N} \delta_h\} \tag{2}$$

This shows a dramatic change from the unobservable Lyapunov function $\mathcal{L}$ to the observable reported demands of bidders and integer price adjustment δ.

The right-hand max-min formula says that when the auctioneer adjusts the prices from $p(t)$ to $p(t+1) = p(t) + \delta(t)$, she tries to balance two opposing forces by minimizing every bidder's loss for every possible price change δ in $\mathcal{D}$ and choosing one price change that maximizes the seller's gain from all possible price changes.

In the auction process bidders do nothing but report their demand sets

The universally convergent dynamic (UCD) auction

Step 1: The auctioneer announces an (arbitrary) initial price vector $p(0) \in \mathbb{Z}^N$. Let $t := 0$ and go to Step 2.

Step 2: Every agent $j \in B_0$ reports her demand $D^j(p(t))$ at $p(t)$ to the auctioneer. Then based on reported demands $D^j(p(t))$, the auctioneer finds an integer solution $\delta(t)$ to the right side problem of (2). If the zero vector $\delta(t) = 0$ is an optimal solution, the auction stops. Otherwise, the auctioneer adjusts prices by setting $p(t+1) := p(t) + \delta(t)$ and $t := t + 1$. Return to Step 2.

Theorem 3: Under Assumptions (A1) and (A2), starting with any given initial price vector $p(0) \in \mathbb{Z}^N$, the UCD auction finds an integer competitive equilibrium vector in a finite number of rounds.

Recall Example 2 with two items $A = (1, 0)$ and $B = (0, 1)$. Every agent knows her values privately. Agents' values are given in Table 1.

Table 1: *Valuations of Agents over Bundles.*

Agents\Bundles	$\emptyset$	$A = (1, 0)$	$B = (0, 1)$	$AB = (1, 1)$
Bidder 1	*0*	*2*	*2*	*5*
Bidder 2	*0*	*2*	*2*	*5*
Bidder 3	*0*	*1*	*1*	*4*
Seller	*0*	*1*	*1*	*3*

For this example, we have the demand type $\mathcal{D} = \{\pm(1, 0), \pm(0, 1), \pm(1, 1)\}$ and its search set $\mathcal{SD} = \{(0, 0), \pm(1, 0), \pm(0, 1), \pm(1, -1)\}$.

Let us first see how a multi-item English auction would operate for this example. The seller initially announces low prices $p(0) = (p_A(0), p_B(0)) = (0, 0)$. Clearly, every agent demands the two items. As the bundle AB is overdemanded, the auction will raise the two prices simultaneously, say each by one unit, an integer increment as a typical English auction does. The price vector is updated to $p(1) = (1, 1)$. At $p(1)$, AB is still overdemanded and the prices are raised up to $p(2) = (2, 2)$. At $p(2)$, AB is still overdemanded and the price is updated to $p(3) = (3, 3)$. At $p(3)$ no bidder wants to demand any item and the auction has stuck in a non-equilibrium state. This phenomenon is called the exposure problem; see e.g., Milgrom (2000).

Let us see how our basic auction resolves the exposure problem. At any time $t \in \mathbb{Z}_+$, in order to reduce the value of the Lyapunov function, the auctioneer adjusts the current prices $p(t)$ to the next prices $p(t+1) = p(t) + \delta(t)$ by finding an optimal search direction $\delta(t)$ to the following problem until the vector of zeros becomes an optimal solution:

$$\max_{\delta \in \mathcal{SD}} \{ \sum_{j \in B_0} \min_{x^j \in D^j(p(t))} x^j \cdot \delta - \sum_{i \in N} \delta_i \} \tag{3}$$

Recall that the search set $\mathcal{SD} = \{(0,0), \pm(1,0), \pm(0,1), \pm(1,-1)\}$. Starting with prices $p(0) = (p_A(0), p_B(0)) = (0,0)$, the auctioneer updates prices $p(t+1) = p(t) + \delta(t)$ according to (3). At $p(0)$, the bundle AB is demanded by every agent. There are two optimal adjustments $(1,0)$ and $(0,1)$ and we can choose either of the two. The process is shown in Table 2. The auction stops at $p(5) = (3,2)$ and finds a WE in which AB is given to bidder 1 or 2 who pays 5, and other bidders get nothing and pay nothing. The auction can also stop at $(2,3)$ if one chooses $\delta(4) = (0,1)$ at $p(4) = (2,2)$.

Table 2: *Illustration of the Basic Auction.*

time t	*prices* $p(t)$	$\delta(t)$	$D^0(p(t))$	$D^1(p(t)) = D^2(p(t))$	$D^3(p(t))$
0	$(0,0)$	$(1,0)$	$\{AB\}$	$\{AB\}$	$\{AB\}$
1	$(1,0)$	$(0,1)$	$\{AB\}$	$\{AB\}$	$\{AB\}$
2	$(1,1)$	$(1,0)$	$\{AB\}$	$\{AB\}$	$\{AB\}$
3	$(2,1)$	$(0,1)$	$\{AB, B, \emptyset\}$	$\{AB\}$	$\{AB\}$
4	$(2,2)$	$(1,0)$	$\{\emptyset\}$	$\{AB\}$	$\{AB, \emptyset\}$
5	$(3,2)$	$(0,0)$	$\{\emptyset\}$	$\{AB, B, \emptyset\}$	$\{\emptyset\}$

Agent $j \in B_0$ *bids sincerely* with respect to her utility function u^j if she always submits a bid $B^j(t)$ equal to her demand set $D^j(p(t)) = \arg\max_{x \in \{0,1\}^N} \{u^j(x) - p(t) \cdot x\}$ at every time $t \in \mathbb{Z}_+$ and any price vector $p(t) \in \mathbb{R}^N$.

Recall that $\mathcal{M}$ stands for the (original) market with m bidders and the seller with the set N of n items. For every bidder $j \in B$, let $\mathcal{M}_{-j}$ denote the market $\mathcal{M}$ *without the participation of bidder* j and $B_{-j} = B_0 \setminus \{j\}$. Let $\mathcal{M}_{-0} = \mathcal{M}$ and $B_{-0} = B_0$. So, for every $k \in B_0$, the sub-market $\mathcal{M}_{-k}$ comprises the set B_{-k} of agents and the set N of n items.

Incentive Compatible (IC) Auction: Basic Idea

The basic idea of the IC dynamic auction to implement the universally convergent dynamic (UCD) auction for every sub-market $\mathcal{M}_{-k}$ ($k \in B_0$) simultaneously from the same starting price vector. This will create $m + 1$ paths of price vectors. By using the bids of every bidder and the generated price vectors the auction will generate a VCG outcome and give every bidder a net profit equal to his marginal contribution.

It will be shown that bidding sincerely is an ex post perfect Nash equilibrium.

The IC dynamic auction runs the universally convergent dynamic (UCD) auction for all markets $\mathcal{M}_{-k}$ ($k \in B_0$) simultaneously with the following modifications:

Let $p^k(t)$ denote the prices of each market $\mathcal{M}_{-k}$ ($k \in B_0$) at time $t \in \mathbb{Z}_+$. Then at $t \in \mathbb{Z}_+$ and with respect to $p^k(t) \in \mathbb{Z}^N$, every bidder $j \in B_{-k}$ submits his bid $B_k^j(t) \subseteq \{0,1\}^N$ which may differ from his true demand set $D^j(p^k(t))$, but the seller's bid $B_k^0(t)$ always equals her true demand set $D^0(p^k(t))$. The auctioneer solves the following decision problem, i.e., the problem (2)

$$\max_{\delta \in \mathcal{SD}} \{ \sum_{j \in B_{-k}} \min_{x^j \in B_k^j(t)} x^j \cdot \delta - \sum_{h \in N} \delta_h \} \tag{4}$$

When the price adjustment $\delta^k(t)$ being a solution to (4) is equal to the vector of zeros, it means that the auction finds an "equilibrium allocation" $X^k = (x^{k,j}, j \in B_{-k})$ in the market $\mathcal{M}_{-k}$ in the sense that $x^{k,j} \in B_k^j(t)$ for every $j \in B_{-k}$ and $\sum_{j \in B_{-k}} x^{k,j} = \sum_{h \in N} e(h)$. As long

The Incentive Compatible Universal Dynamic (ICUD) Auction: Part I

Step 1: At the start, the auctioneer announces a common price vector $p^k(0) = p(0) \in \mathbb{Z}^N$ for all markets $\mathcal{M}_{-k}$, $k \in B_0$. Let $t := 0$ and go to Step 2.

Step 2: At each time $t \in \mathbb{Z}_+$ and prices $p^k(t) \in \mathbb{Z}^N$, every agent $j \in B_{-k}$ submits her bid $B_k^j(t) \subseteq \{0,1\}^N$. Based on reported bids, if the auctioneer finds an equilibrium allocation X^k in any market $\mathcal{M}_{-k}$ at the current round, she records the current prices as $p^k(T^k) \in \mathbb{Z}^n$ and the current time as $T^k \in \mathbb{Z}_+$. For any other market $\mathcal{M}_{-k}$ which is not in equilibrium, the auctioneer calculates a price change $\delta^k(t)$ according to (4) and announces a new price vector $p^k(t+1) = p^k(t) + \delta^k(t)$ for the market $\mathcal{M}_{-k}$. The UCD auction goes back to Step 2 with $t := t + 1$. When the auction has

The Incentive Compatible Universal Dynamic (ICUD) Auction: Part II

Step 3: All markets now clear. For every market $k \in B_0$ and every agent $j \in B_{-k}$ at every time $t = 0, 1, \cdots, T^k - 1$, based on her reported bids $B_k^j(t)$ and the price change $\delta^k(t)$ the auctioneer calculates agent $j's$ 'indirect utility reduction' $\Delta_j^k(t)$ when prices are changed from $p^k(t)$ to $p^k(t+1)$ in the market $\mathcal{M}_{-k}$, where

$$\Delta_j^k(t) = \min_{x^j(t) \in B_k^j(t)} x^j(t) \cdot \delta^k(t) \tag{5}$$

Every bidder $j \in B$ will be assigned the bundle $x^{0,j}$ of the allocation $X^0 = (x^{0,j}, j \in B_0)$ found in the original market $\mathcal{M}_{-0} = \mathcal{M}$ and asked to pay β_j, with the option to decline and walk away, when his payoff becomes negative, where

$$\beta_j = \sum_{h \in B_{-j}} \left[\left(\sum_{t=0}^{T^0-1} \Delta_h^0(t) - \sum_{t=0}^{T^j-1} \Delta_h^j(t) \right) + x^{j,h} \cdot p^j(T^j) - x^{0,h} \cdot p^0(T^0) \right] \tag{6}$$

The Incentive Compatible Universal Dynamic (ICUD) Auction: Part III

Step 4: In this case the auction does not find an allocation in every market $\mathcal{M}_{-k}$, $k \in B_0$. In the end, every bidder $j \in B$ gets nothing and pays nothing.

The payment formula β_j uses only revealed information and is simple and easy to calculate and equals the first term Plus the second term and Minus the third term.

The first term is the accumulation of 'indirect utility reduction' of bidder $j's$ all opponents in B_{-j} along the path from $p^j(T^j)$ to $p(0)$ in the market $\mathcal{M}_{-j}$ and along the path from $p(0)$ to $p^0(T^0)$ in the market $\mathcal{M}$;

The second term stands for the total equilibrium payment by all

In the literature for static auction games of incomplete information, the notion of *ex post Nash equilibrium* has been used by Cremer and McLean (1985) and Krishna (2002). This solution requires that the strategy for every player, i.e., bidder, should remain optimal if the player were to get to know types of his opponents.

Ausubel (2004, 2006) and Sun and Yang (2014) have adopted the solution of *ex post perfect Nash equilibrium* to dynamic auction games of incomplete information which requires the same condition for every player at every node of their studied dynamic auction games.

Every bidder's bidding strategies depend on his own valuation function, his own bidding history and all other information (prices and bids etc) he has known or observed so far. On and off equilibrium paths need to be considered.

Theorem 4: Suppose that the market $\mathcal{M}$ satisfies Assumptions (A1)

A mechanism is *beneficial to every agent* if the payment the seller receives for every sold bundle is at least as big as her reserve value of the bundle or the total utility she receives is at least as good as she does not trade, and if the net profit for every bidder is nonnegative.

A mechanism is said to be *ex post individually rational*, if, for every bidder, no matter how his opposing bidders act in the auction, as long as he is sufficiently able to judge whether his payoff is negative or nonnegative, he will never end up with a negative payoff.

Theorem 5: Suppose that the market $\mathcal{M}$ satisfies Assumptions (A1) and (A2).
(1) If every bidder acts truthfully, the ICUD auction is beneficial to every agent.
(2) The ICUD auction is ex post individually rational.

Thank You!