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POPPER AND THE "ABSOLUTE PROOFS"

In some texts written during the last years of his life, Popper suggests that the "axiomatic method", the method of making a few assumptions and then deducing from them, by purely logical means, the huge edifice of theorems, is not the only possible method in mathematics. Where this axiomatic method is adopted, there is no room for "absolute proofs": any supposed proof is relative to some propositions which cannot be proved. But according to Popper, pre-Euclidian Greek mathematics contained valid proofs without assumptions, thus absolute proofs. This fact, he adds, has been forgotten after Euclid and the success of his axiomatic system in the ***Elements***. So, an interesting area of mathematics has been fatally neglected.

Let us consider the expression "a proof without assumption". We may remark, first, that it is impossible to conceive a statement, and especially a proof, which would be completely "without assumption". Thus we must suppose that in Popper's mind a proof may be described as being "absolute" although it depends on many assumptions, provided that this proof is "without assumption" in a certain meaning which we have a good reason to regard as the only pertinent meaning. Our second remark is that the alleged absolute proof must be "without assumption" in an explicit way, openly. For the simple fact that no assumption is explicitly stated is not sufficient to conclude that a proof is without assumption. Searching after implicit assumptions is precisely the main task of axiomatic method. Popper is required to convince us that pre-Euclidian proofs were not simply incomplete or pre-mathematical proofs, whose authors had not yet undertaken to clear up their assumptions.

He gives two reasons to establish that it is not so, that these proofs were without assumption in a pertinent meaning and in an explicit way: two different reasons, concerning two different kinds of proofs. According to the first reason, a proof was without assumption because it was a *reductio ad absurdum*. According to the second reason, a proof was without assumption because it was intuitive.

1. The first kind of absolute proofs: proofs by *reductio ad absurdum*

The best exemplification of the first reason is the proof, probably discovered within the Pythagorean school, that the diagonal *a* of a square is incommensurable with the side *b* of this square. The proof was clearly a *reductio ad absurdum*: supposing that the diagonal is commensurable with the side, it was proved that in the ratio a / b the denominator b has to be both odd and even: the supposition leads to an absurd conclusion.

Was this proof an absolute proof? At first sight, the reason given by Popper does not seem to be a very convincing one. Everybody knows that Euclid's ***Elements*** contain many proofs *by reductio ad absurdum*, which are as relative to the axioms as the other proofs. For a *reductio ad absurdum*, one could say, is nothing but an indirect means to deduce the proposition which ought to be deduced from the axioms, when this proposition could not be deduced directly. Thus a false supposition is voluntarily and cunningly tried in order to prove, through the absurdity it entails, that the opposite is needed.

But according to Popper, the case of the pre-Euclidian proof of the incommensurability of the diagonal was totally different. This proof was not an indirect proof for lack of a direct one. It was a genuine refutation (*elenchus*), the falsification of a conjecture: namely the Pythagorean conjecture that all measurement must be counting a certain number of natural units.

Accordingly, the proof of the incommensurability of the diagonal was clearly "without assumption" in an explicit way, since it destroyed its own assumption that there should be two natural numbers whose ratio equals $\sqrt{2}$. Admittedly, many other assumptions were not destroyed, but they did not matter: the only pertinent assumption was the one which was testing itself through the proof, and thereby refuted by it. Then, the proof must be said "absolute" because of its negative meaning: in view of a logical asymmetry, the confirmations of a conjecture are always relative to that conjecture, but when it is refuted we are facing up to the absolute.

Such a proof, Popper claims, was not a pre-mathematical one. Rather it belonged to a forgotten mathematical area. Early Greek mathematics was a science of the world, a cosmology. It was not divided into branches like arithmetic or geometry, but into competing theories like arithmetization (Pythagoras) or geometrization (Plato). Its assumptions were not axioms, supposed to convey their evidence to all the theorems, but conjectures, liable to be refuted if one of their consequences proved to be false.

2. The second kind of absolute proofs: intuitive proofs

What Popper says about *reductio ad absurdum* seems to imply that only negative proofs might be absolute proofs. Yet he asserts that some pre-Euclidian proofs were "without assumption" for a different reason: because of their "intuitive" character. The examples he gives point out that this new reason concerned especially geometrical proofs. The best example is Plato's proof, in the ***Meno***, that the square over the diagonal of any given square has twice the area of that given square. It was a diagrammatic proof, which consisted in drawing a square with one diagonal, in extending the drawing to the square over this diagonal, and then showing to a young boy, unlearned in geometry, that the latter square contains four isosceles rectangular triangles equal to the two isosceles rectangular triangles contained in the former square.

The intuitive character of this proof is unquestionable. But compared with the Euclidian demonstrations on the same subject, this intuitive character may be suspected of being a symptom of incompleteness rather than a mark of "absoluteness": for instance, Plato did not take trouble to prove the equality of the isosceles rectangular triangles. Moreover, Popper has continuously said, in his whole work, that we do have, admittedly, something like intuitions, but that intuition is not an infallible criterion of truth: how could it be a basis for a proof?

What is properly intuitive, in Plato's proof, is some acquaintance with geometrical figures, especially with the square and the isosceles rectangular triangle. What the proof exactly proved, according to Popper, is that this acquaintance is sufficient for solving a problem of measurement. Nothing else is needed, especially counting (or only counting up to 4!). Admittedly, the Pythagorean assumption that all measurement is ultimately counting of natural units had already been refuted by the proof of the incommensurability of the diagonal. But Plato's proof, through its intuitive character, taught the converse truth: it taught that this false assumption, anyway, is a needless assumption, since we can measure anything without counting. So, nothing but the consideration of geometrical figures is required to understand geometrical figures. We have not to decompose them in their elements: geometrical figures themselves are the elements. In pre-Euclidian geometry, appealing to intuition was not directed against logic: it was directed against arithmetic.

Thus, Plato's proof was without assumption in the sense that it needed no arithmetical assumption. Of course, this proof needed many other assumptions. What matters, however, is that it could work as a geometrical proof only by rejecting as needless any arithmetical assumption of commensurability or rationality: here was the pertinent meaning of "without assumption". The intuitive character of the proof made this "without" an explicit one.

The first example of absolute proof established the impossibility of an arithmetical (Pythagorean) cosmology. The second example suggested that this impossibility did not prevent the foundation of a cosmology, since what was impossible was at the same time needless. A geometrical cosmology, a cosmology whose elements are the shapes or figures themselves, was therefore possible.

3. The absolute proofs forgotten

Popper asserts that the treatise of Euclid's ***Elements*** was originally intended as an attempt to solve systematically the problems raised by this geometrical cosmology, and not as the textbook of pure geometry it has been supposed afterwards. Popper explains this change in the interpretation by the remarkable success of Euclid in solving its problems. For he succeeded in working out a completely autonomous geometry, freed from any arithmetical assumption of commensurability or rationality, and thereby protected against incommensurability and irrationality. But this geometry was so completely autonomous that people might well forget in what respect it was autonomous, and consider it as pure geometry. Solving the problems had removed the memory of these problems.

When this pure geometry became a simple branch of mathematics, side by side with an arithmetical branch in which irrationals could be accepted as a peculiar kind of numbers, the reason why arithmetization had been previously rejected, and geometrization preferred, loose its significance. But this reason was also, precisely, the reason why pre-Euclidian mathematicians could offer absolute proofs, proofs without (arithmetical) assumption, either because this assumption was proved to be false, or because it was proved to be needless. Then, it was inevitable that the absoluteness of these proofs would be forgotten. From a purely geometrical point of view, it does not make any sense to neglect as irrelevant most of the assumptions of a proof, and only regard as pertinent those which may be proved to be false or needless. All the assumptions, now, are preserved from the risk of being refuted; all are equally pertinent in order to deduce the theorems; all must be equally stated in an explicit way.

Axiomatic geometry has superseded in that way the conjectural method of mathematical cosmology. As a consequence, pre-Euclidian's mathematical proofs have not only lost the reason which made them absolute proofs, proofs without assumption. Compared with the new standard of axiomatic method, they could not fail to appear in retrospect as incomplete proofs whose assumptions were not yet explicitly stated.

What happened, lastly, about the division of absolute proofs between proofs by *reductio ad absurdum* and intuitive proofs? In pre-Euclidian mathematics, these two kinds of proofs were united by their complementarity: each of them was needed to reject the pertinent assumption in a specific way. In contrast, they become regarded after Euclid as two conflicting kinds of proofs: using *reductio ad absurdum* attest a blind reliance on logic and its law of excluded middle, while appealing to intuition means that mathematics is not reducible to logic.