Non-normal Interpretations of Positive Logic

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This presentation analyses a problem raised by Popper in connection to Carnap's 1943 book, *Formalization of Logic*, in a private letter from that year, namely, whether the calculus of positive propositional logic allows for non-normal interpretations. Before considering this problem and in order to understand its relevance, we have to explain **a)** why Popper is interested in positive logic and **b)** why it is important to study the non-normal interpretations in connection to it.

a) *Popper's interest in positive logic*

[Popper 1947a: 290] emphasizes that the distinction between derivation and demonstration (or proof), with some exceptions, "has been often neglected by logicians". Among the exceptions, [Carnap 1937: 28-29, 1942: 167] takes a demonstration to be a special case of derivation, namely, that derivation in which the conclusion "is derivable from the null series of premises, and hence from any sentence whatsoever". In agreement with Carnap (see Popper 1947a, footnote 24), Popper defines a proof as that derivation which "asserts the truth of the conclusion absolutely – independently of the question whether any particular other statement is true". Thus, in a proof, "the conclusion can be validly derived from any premise whatsoever" [Popper 1947b: 231]. The main idea is that a proved statement is true independently of the truth of the premises from which it is derived, while in a regular derivation the conclusion is true provided that the premises are true.

The distinction between derivation and demonstration underlies two ways of constructing a system of logic: as a derivational logic or as a demonstrational logic. A system of logic "intended from the start to be a theory of inference in the sense that it allows us to derive from certain informative (non-logical) statements other informative statements" [Popper 1947b: 230] is *a derivational logic*, i.e., it contains rules of inference for drawing consequences from hypotheses. In contrast, "most systems of modern logic are not purely derivational, and some (for example in the case of Hilbert Ackerman) are not derivational at all." [ibid.] These systems are *demonstrational logics*. The derivations conducted in them *usually* start from logical axioms, definitions or theorems. However, this is not always so because, for instance, in an indirect proof (*reductio ad absurdum*) the premises are jointly contradictory. The main point is that even in an indirect proof, as in any proof, the truth of the conclusion holds independently of the truth of the premises.

[Popper 1970: 17-20] treats the two systems of logic in connection to two essential features of deduction, i.e., the transmission of truth and the retransmission of falsity, and talks about two uses of logic: *a demonstrative use*, in the mathematical sciences, for proofs, and *a derivational use*, in the empirical sciences, for criticism. In critical discussions in general, and in empirical sciences in particular, we should use the strongest logic, i.e., classical logic, because we want our criticism to be severe.[1] However, in the mathematical sciences, we should use a *minimum apparatus* instead of any strong logical means. Popper regards a proof of a known theorem that deploys weaker resources than the old proof as 'a real mathematical discovery'. This is so because what we aim at in 'sophisticated mathematics' is to know what is *necessary* for proving a theorem

[1] [Popper 1970: 35] mentions that a weakening of classical logic, like that suggested by Birkhoff and von Neumann, or by Reichenbach, is not adequate in the empirical sciences, because it can render an empirical theory irrefutable.

and not only what is sufficient. Therefore, in the demonstrative use, we should weaken the classical logic as much as possible, "and we can, for example, introduce intuitionist logic or some other weaker logic such as positive logic, and investigate how far we can get without using the whole battery." [Popper 1970: 19]. We see thus that for Popper intuitionist logic and in particular positive logic, which is a common sub-system of both classical and intuitionist logics, could serve as a firm foundation of mathematical proofs. However, is positive logic an *objective* instrument for carrying on mathematical proofs? In particular, do the statements of positive logic have a unique meaning or do they allow for non-normal interpretations of their logical constituents?

b) *Non-normal interpretations and positive logic*

In *Formalization of Logic,* Carnap proved that the standard formalizations of classical propositional and predicate logic allow for non-normal interpretations, i.e., interpretations for which the calculi remain sound and complete, but in which the logical constants have different meanings than the standard ones. For instance, there are non-normal interpretations in which a sentence and its negation are both true and non-normal interpretations in which a disjunction is true although both of its disjuncts are false. The existence of such interpretations shows that the standard calculi do not fully formalize all the logical properties of the logical terms and, thus, fail in uniquely determining their meaning.

Carnap's discovery of the non-normal interpretations is seen nowadays as a challenge to logical inferentialism[2] (i.e., the view that the formal rules of inference uniquely determine the meaning of the logical terms). It is surprising then that, although Popper knew of Carnap's results, he defined, some years later, the logical constants in inferential terms. In his review of Popper's article, "Logic without Assumptions", J. McKinsey pointed out correctly, though without referring to Carnap's results, that Popper's inferential definition of disjunction is inadequate, since it may lead to the result that a sentence follows from a disjunction although it follows from neither of its disjuncts.

In the above-mentioned letter, from July 5th 1943, Popper wrote to Carnap that he also believed that the truth-tables are not fully formalized by the propositional calculus, but had no idea how this problem could be spelled out. Fascinated by Carnap's existence proof for the non-normal interpretations, Popper went further and asked whether a specific sub-system of propositional logic, namely, the positive propositional logic (i.e., propositional logic without negation, formulated by Hilbert and Bernays) allows for non-normal interpretations. More precisely, Popper wondered whether:

- **I)** the axioms of positive logic allow non-normal interpretations in general, and for implication in particular;
- **II)** by adding the axioms for conjunction and equivalence to the implicational axioms of positive logic, the new system allows non-normal interpretations for conjunction and implication, and
- **III)** what happens if we add, separately, to the system of positive logic defined at (II), the axioms for disjunction and the axioms for negation.

[2]See Raatikainen 2008, Murzi & Hjortland 2009, Bonnay, D. and Westerståhl, D. 2016.

In his response to Popper, from December 9th 1944, Carnap qualified Popper's remarks as "the best comments I have received on this book" and confessed to Popper that he had not given much study to the 'positive logic' and, thus, did not know whether there are non-normal interpretations for these systems. However, he encouraged Popper to investigate the problem and told him that "if you find any results, they should be published in the *Journal of Symbolic Logic*." Although Popper did not investigate this problem any further, his questions deserve attention, especially given the important role ascribed by him to positive logic in proofs.

In this presentation, we answer some of Popper's questions regarding the existence of non-normal interpretations for the systems described under I-III, and then discuss some of their consequences for his distinction between demonstration and derivation. We consider the relationship between the existence of non-normal interpretations of a logical system and thus its failure to determine uniquely the meaning of logical terms, on the one hand, and its construction as a derivational or a demonstrational logic, on the other hand. In particular, we discuss the question whether a system of logic that admits of non-normal interpretations could satisfy Popper's constraints on mathematical demonstration, i.e., his insistence that it ought to use a minimal logic apparatus.

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