The optimal prize structure of symmetric Tullock contests*

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Abstract

We show that the optimal prize structure of symmetric n-player Tullock tournaments assigns the entire prize pool to the winner, provided that a symmetric pure strategy equilibrium exists. If such an equilibrium fails to exist under the winner-take-all structure, we construct the optimal prize structure which improves existence conditions by dampening efforts. If no such optimal equilibrium exists, no symmetric pure strategy equilibrium induces positive efforts. (JEL *C7, D72, J31.* Keywords: *Tournaments, Incentive structures, Rent seeking.*)

1 Introduction

It is well known that "an income maximizing contest administrator obtains the most rent-seeking contributions when he makes available a single, large prize" Clark and Riis (1998b). Less, however, is known about effort maximizing prizes in Tullock contests when an equilibrium supporting this winner-take-all structure does not exist. The same is true when a generalized rent-seeking context is considered, i.e., if non-linear costs are allowed to accompany the outlays of more than two contestants. Unfortunately, both cases arise typically in the popular applications of the theory such as lobbying activities, research and development races, or competitions for promotions, to name but a few.

We show that with symmetric players the winner-take-all prize structure induces maximal efforts regardless of the number of players or their effort cost, provided that a symmetric pure strategy equilibrium (SPSE) exists. In cases where such an equilibrium fails to exist under the winner-take-all

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prize structure, we improve existence conditions for SPSE by allowing for multiple prizes.[1] These necessary and sufficient conditions for existence depend on the number of players in a fashion such that non-existence can be overcome by adding more players. We identify the optimal, i.e., effort maximizing, prize structure among all structures for which SPSE exist and provide a procedure for finding such a structure whenever a SPSE exists. The driving force behind our results is, quite counter-intuitively, that multiple prizes reduce incentives for effort provision in the class of tournaments considered.

Besides the above mentioned Clark and Riis (1998b), the paper closest to our analysis is Barut and Kovenock (1998) who analyze the fully discriminating, complete information all-pay auction (the limiting case of our setup). In stark contrast to our results, they derive the near total arbitrariness of prize structures in their perfectly discriminatory setup where effort is the only determinant of winning. Adding incomplete information to this all-pay auction setup, Moldovanu and Sela (2001) show that more than one prize is optimal when contestants have convex costs. It may thus come as a surprise that our optimal prize structures are independent of the curvature of costs. The reason for this disparity is that their heterogenous players have private effort costs which affect bidding behavior. Intuitively, a second prize in an asymmetric contest can be desirable for the maximization of total efforts because a single first prize may undermine the incentives of both weak contestants expecting not to win and of strong contestants believing to be able to win with little effort.[2]

We concentrate on SPSE because very little is known about the properties of mixed strategy equilibria for imperfectly discriminatory success functions—particularly when existence of SPSE fails. We know of two results only: Baye, Kovenock, and de Vries (1994) show that mixed equilibria exist for the two-players case, can be solved for in principle, and provide a handful of discrete-grid examples. Alcalde and Dahm (2010) use a similar technique to address the (generalized) problem and show that in their mixed strategy equilibrium rents are fully dissipated. Neither result can be used to analyze mixed equilibria in our setup. More importantly, from a contest design point of view, it is not clear what revenue a mixed equilibrium generates in a one-shot game. As observed by Baye, Kovenock, and de Vries (1999)—although in expectation the complete rent is dissipated in such a mixed equilibrium, i.e., the contest extracts the highest possible total expected effort level from the contestants—the effort actually extracted depends on the realization of the random strategy profile and is thus random itself. Therefore, a designer who runs the contest just once would certainty have good arguments to attempt the implementation of the optimal effort, pure strategy equilibrium suggested in the present analysis. This observation is confirmed by the absence of the near-arbitrary prize structure variations predicted by the mixed strategy equilibrium in actually played (e.g., lottery) contests. Models featuring asymmetric pure strategy equilibria are presented by Cornes and Hartley (2005), Szymanski and Valletti (2005), and Siegel (2009). We suspect that solving for the optimal

[1] An alternative interpretation of our optimal prize structure is a vector of rank-dependent compensation payments designed to induce maximal efforts.

[2] Indirectly related studies concerned with multi-stage optimal prizing are Gershkov and Perry (2009) and Fu and Lu (2009). The former analyze the effort effects of introducing midterm reviews into the contest while the latter show the optimality of a single prize at each stage of their multi-stage Tullock tournament (with linear cost) in a parameter range where existence is not problematic. Franke, Kanzow, Leininger, and Schwartz (2009) maximize total efforts in a rank-seeking environment by attributing a vector of weights to contestants' efforts.

prize structure of an asymmetric version of our model is possible in principle but would be technically challenging. A recent and comprehensive review of the tournaments literature is Konrad (2008). It allows us to to omit all but the most relevant references and keep our literature review brief.

Our contribution shows that multiple prizes do not increase efforts in imperfectly discriminatory Tullock contests. On the contrary, they *dampen* incentives compared with a single prize. We exploit this feature of multi-prize configurations to obtain equilibrium existence when equilibria do not exist when the winner takes all. We provide necessary and sufficient conditions for the existence of SPSE and show how to construct an effort-optimal prize structure whenever SPSE exist. Novel implications of our analysis include that a designer who suspects that a winner-take-all competition may overheat and fail to induce equilibrium efforts among contestants might be able to remedy this problem by either adding further prizes to the contest or by trying to attract more competitors. After introducing the model, all results are presented in the next section followed by a short discussion of their intuition. We stress that versions of proposition 1 and 2 are well known although our proofs are new. Our formulations only add the—to our knowledge—novel observation that SPSEs are possible with concave cost functions. The reason why we (re)state these propositions is that our principal and new results 3 and 4 rely on the arguments developed in the proofs of proposition 1 and 2. These and all other proofs can be found in the appendix.

2 Model and results

We consider a set $\mathcal{N}$ of $n > 1$ symmetric, risk neutral players engaging in a contest where any player $i \in \mathcal{N}$ exerts effort $e_i \in [0, \infty)$. There is a fixed prize pool $P > 0$ from which prizes $P^1, P^2, \ldots, P^n$, $P = \sum_l P^l$, awarded to the contest winner, second place, an so on are taken. The contest satisfies limited liability and the designer sets $P^l \geq 0$, $l = 1, \ldots, n$ in order to maximize the sum of efforts. Denote the vector of all players' efforts by $\hat{e} = (e_1, e_2, \ldots, e_n)$. Then the winning probability of player i exerting effort e_i with her opponents choosing $\hat{e}_{-i}$ is given by the Tullock success function as[3]

$$f_i^1(\hat{e}) = \frac{e_i^r}{\sum_{j \in \mathcal{N}} e_j^r} \text{ for } r > 0.$$

We define $f_i^1(0) = 1/n$ for completeness. The probabilities of winning the second and subsequent prizes $f_i^2, f_i^3, \ldots$ are given by the nested Tullock success function, i.e., by recursively applying the above success function to the set of players while eliminating the winners of the previous stages. Hence player i chooses her effort e_i in order to maximize her utility

$$\arg\max_{e_i} \sum_{l=1}^{n} \left(f_i^l(e_i, \hat{e}_{-i}) P^l \right) - c(e_i) \tag{1}$$

[3] Skaperdas (1996) argues that the Tullock form is less special than one might believe. In particular, he shows that it is the only ratio-based function fulfilling a small set of intuitive desiderata. Fu and Lu (2007) and Jia (2008) derive distribution-based foundations for the general Tullock formulation.

where we assume $c(e_i)$ to be strictly increasing, differentiable, and satisfying $c(0) = 0$. Moreover, we assume that $c'(e)e$ is a strictly increasing function of e.[4] Assuming the existence of SPSE for any given prize structure, our first result shows that the winner-take-all structure induces the greatest efforts for arbitrary costs.

Proposition 1. *(i) Under our assumptions, if SPSE exists for a given prize structure, then it is unique. (ii) If SPSE exists for any prize structure then the Tullock tournament which induces the largest sum of equilibrium efforts from symmetric contestants assigns the whole prize pool to the winner.*

A corollary of this result, using the same proof, is that an optimal prize structure must be monotonic in the sense of $P^1 \geq P^2 \geq \cdots \geq P^n$. The second proposition generalizes the linear cost result by Clark and Riis (1998b) in deriving a SPSE existence condition for the winner-take-all prize structure.[5] From now on, we restrict attention to cost functions of the form $c(e) = ae^b$ with $a, b > 0$.

Proposition 2. *Existence of SPSE under the winner-take-all prize structure $P^1 = P$ is ensured if and only if*

$$\frac{r}{b} \leq \frac{n}{n-1}. \tag{2}$$

If condition (2) does not hold, then the existence of SPSE fails under the winner-take-all prize structure because, when all players adhere to the effort level of the unique SPSE-candidate, their utility decreases below their zero-effort utility. Hence this prize structure causes excessive efforts in the sense of the efforts being too costly to ensure the players' participation and therefore destroys the equilibrium.

The case of the 'lottery' contest in which a single effort vector $(e_i, \hat{e}_{-i})$ determines the winning probabilities for a whole index set of prizes $\mathcal{S}$ with $s = |S| \leq n$ is similar. In this type of contest, player i's objective is

$$\arg\max_{e_i} \sum_{l \in \mathcal{S}} f_i^1(e_i, \hat{e}_{-i}) P^l - c(e_i),$$

i.e., there are s sequential, independent prize draws made for the top-ranked player from $(e_i, \hat{e}_{-i})$ to award s prizes. Since $\sum_{l \in \mathcal{S}} f_i^1(e_i, \hat{e}_{-i}) P^l = f_i^1(e_i, \hat{e}_{-i}) P^{\mathcal{S}}$, $P^{\mathcal{S}} = \sum_{l \in \mathcal{S}} P^l \leq P$, in terms of incentives and existence, this contest equals the winner-take-all setup provided that the full set of prizes can be won by any single player.

We now analyze the effort-optimal SPSE in cases where under the winner-take-all prize structure $P^1 = P$ no SPSE exists. We show that a more evenly distributed prize structure dampens efforts

[4] This assumption ensures the uniqueness of SPSE, whenever it exists. Note that it holds for any convex cost function and many concave cost functions as well. For example, it holds for any monomial cost function of the form $c(e) = ae^b$ with $a, b > 0$. It does rule out, however, certain concave cost functions as, for instance, $c(e) = 1 - \exp(-e)$.

[5] As demonstrated by Szidarovszky and Okuguchi (1997), one can alternatively rescale efforts applying the inverse cost function and then study a linear cost ('rent-seeking') contest. That would, however, change the model primitives contradicting our contest design approach. Moreover, the approach we take makes new economic applications possible. By setting $b < 1$, for instance, we allow for concave cost functions which—to our knowledge—were not previously analyzed in the context of Tullock tournaments.

and extends the range of parameters where existence can be obtained. This insight (and its precise implications) is the principal novel result of this paper.

Proposition 3. *There is a monotonic prize structure for which SPSE inducing positive efforts exist, if and only if*[6]

$$\frac{r}{b} \leq \frac{n-1}{\sum_{m=2}^{n} \frac{1}{m}}. \tag{3}$$

The corresponding condition for rent-seeking games just sets $b = 1$ in (3). Contrary to the previously known existence threshold of $r \leq \frac{n-1}{n}$ for single prize contests, the right-hand side of (3) is increasing in the number of players n, so adding players to the contest improves existence conditions (without bound). In the proof of proposition 3 we show that when condition (3) holds, then there exists a specific prize structure for which a SPSE exists. Note that (3) is a condition on the parameters of the contest and not on the prize structure. Given that this condition is satisfied, we now proceed to identify the optimal prize structure ensuring both SPSE existence and highest equilibrium efforts.

The highest possible effort in any SPSE is $e^* = c^{-1}(P/n)$. Given that (3) is satisfied, the designer can implement maximal equilibrium efforts e^* by using the following simple procedure: First try the prize structure $P^1 = P$, then $P^1 = P^2 = P/2$, then $P^1 = P^2 = P^3 = P/3$ and so forth until the resulting efforts $\tilde{e}$ eventually sink below e^*. For the first such uniform prize structure (s)he then shifts some $\varepsilon > 0$ away from the last prize k and subdivides it equally among the $k-1$ prior prizes until the efforts $\tilde{e}$ exactly equal e^*. The following proposition formalizes this idea.

Proposition 4. *For $n \geq 3$, if (2) is violated but (3) holds, then there exists an integer $2 \leq k < n$ and a real number $0 \leq \varepsilon < \frac{1}{k}P$ such that the prize structure*

$$(\underbrace{\frac{1}{k}P + \frac{1}{k-1}\varepsilon, \frac{1}{k}P + \frac{1}{k-1}\varepsilon, \ldots, \frac{1}{k}P + \frac{1}{k-1}\varepsilon}_{k-1 \text{ times}}, \frac{1}{k}P - \varepsilon, 0, \ldots, 0) \tag{4}$$

is optimal, i.e., induces the highest possible efforts $e^* = c^{-1}(P/n) = \left(\frac{1}{a}\frac{P}{n}\right)^{\frac{1}{b}}$ among all prize structures for which a SPSE exists.

Note that the optimal sum of the players effort $n\left(\frac{1}{a}\frac{P}{n}\right)^{\frac{1}{b}}$ increases in the number of players if the cost function is convex and decreases if the cost function is concave.

3 Discussion

No SPSE inducing positive efforts exists for monotonic prizes if (3) fails. The following picture illustrates the interplay of the above propositions. It shows the utility $U_i(e_i; e^*)$ of a player unilaterally deviating from e^*. The blue downward sloping curve depicts the locus of possible SPSE utility levels for different values of equilibrium effort. Recall that if $(e^*, \ldots, e^*)$ is an SPSE, then players' utility is given by $P/n - c(e^*)$.

[6] Since (2) and (3) coincide for $n = 2$, existence conditions cannot be improved in this well-known case.

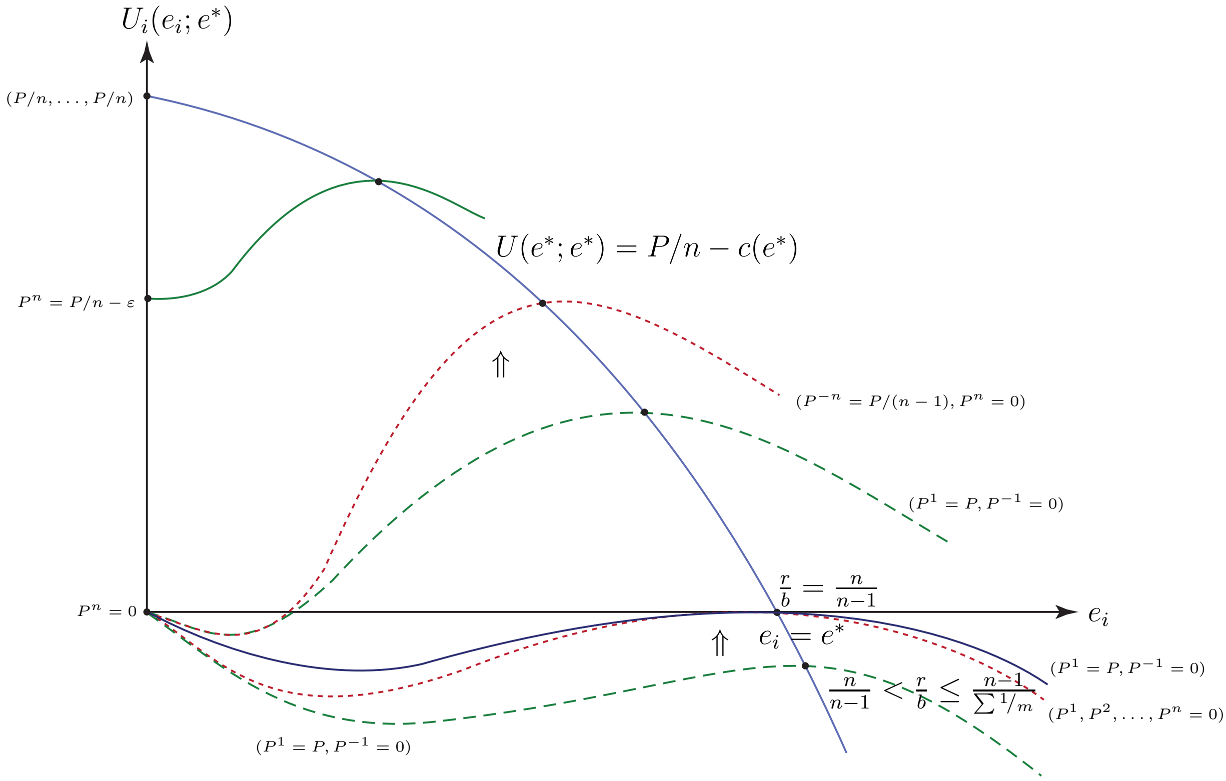

Figure 1: Unilateral deviations from SPSE for different prize structures and values of r/b.

There are two things to note: First, softening incentives by adopting suboptimal multi-prize structures increases the players' equilibrium utility and therefore reduces equilibrium efforts (shifting from the green dashed utility levels to the red dotted utilities). Second, given the parameters of the contest, there are two possible ways to achieve the maximum effort equilibrium. On the one hand, if $\frac{r}{b} = \frac{n}{n-1}$, then the winner-take-all prize structure achieves this equilibrium (the solid blue utility level). On the other hand, if $\frac{n}{n-1} < \frac{r}{b} < \frac{n-1}{\sum_{m=2}^{n} 1/m}$, then there exists an optimal prize structure with more than one prize—described by proposition 4—which achieves the same maximal-effort in SPSE (the red dotted utility level).

If $\frac{r}{b} < \frac{n}{n-1}$, the designer achieves the maximal SPSE efforts by awarding a single prize to the winner. However, if it is possible to control the parameters of the contest then (s)he should strive to reduce the contestant's equilibrium utility by creating a more precise ranking, i.e., by increasing r.[7] If $\frac{r}{b} > \frac{n}{n-1}$, the agent's utility in the SPSE candidate for a single prize is below her zero effort utility. The designer can then increase the agent's equilibrium utility by dampening incentives by offering more than one prize. This multiplicity of prizes is motivated, however, not by optimality as such but solely by existence. Again, if the designer can influence the contest parameters, then (s)he could reduce r by introducing more noise into the success function and thereby increase SPSE efforts. Moreover, the designer can ease the equilibrium existence problem in the case of multiple prizes by attracting additional participants.

[7] If the designer benefits from retaining part of the prize pool and an equilibrium exists, (s)he can balance this prize reduction against less contestants' efforts without affecting existence. The particular balancing will depend on the designer's utility function but the symmetric equilibrium derived above covers many different objectives and welfare criteria.

Finally, the optimal prize structure (4), which enables the designer to collect maximal efforts, consists of no more than three distinct prizes where the highest and lowest prizes may optimally be awarded to multiple players.

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Appendix: Proofs

Proof of proposition 1. Denote the probability of player i winning the l^{th} prize among n agents by $f_i^l(e_1, e_2, \ldots, e_n)$. Denote also by $(\hat{e}/\{j_1, j_2, \ldots, j_{l-1}\})$ a vector of efforts of all players other than $\{j_1, j_2, \ldots, j_{l-1}\}$. Then the probability of player i winning prize $l \geq 2$ is given by

$$\begin{aligned} f_i^l(\cdot) &= \sum_{\{j_1,j_2,\ldots,j_{l-1}\}\subseteq \mathcal{N}\setminus\{i\}} \Pr\begin{pmatrix} i \text{ wins } P^l| \\ j_{l-1} \text{ wins } P^{l-1}, \\ \vdots \\ j_1 \text{ wins } P^1 \end{pmatrix} \Pr\begin{pmatrix} j_{l-1} \text{ wins } P^{l-1}| \\ \vdots \\ j_1 \text{ wins } P^1 \end{pmatrix} \cdots \Pr\left(j_1 \text{ wins } P^1\right) \\ &= \sum_{\{j_1,j_2,\ldots,j_{l-1}\}\subseteq \mathcal{N}\setminus\{i\}} f_i^1\left(\hat{e}/\{j_1, j_2, \ldots, j_{l-1}\}\right) f_{j_{l-1}}^1\left(\hat{e}/\{j_1, j_2, \ldots, j_{l-2}\}\right) \cdots f_{j_1}^1\left(\hat{e}\right) \end{aligned}$$

where the sums are taken over all ordered sets of $l-1$ players different from i. Notice that $f_i^l(\cdot)$ involves only simple Tullock winning probabilities $f_j^1(\cdot)$. Since Player i maximizes (1), SPSE efforts $\hat{e} = (e, e, \ldots, e)$ satisfy the first-order condition

$$c'(e) = \sum_{l=1}^{n} \left(\alpha_l(\hat{e})P^l\right), \ \alpha_l(\hat{e}) = \frac{\partial}{\partial e} f^l(\hat{e}) \tag{5}$$

where we define $\alpha_l(0) = 0$. Coefficients α_1, α_2, and the general α_l are calculated wlog for player 1. The α_l^* are the symmetric equilibrium versions.

P^1: For the first prize $\alpha_1 = \frac{\partial}{\partial e_1} \frac{e_1^r}{\sum e^r} = \frac{e^r r(n-1)e_1^{r-1}}{\left((n-1)e^r + e_1^r\right)^2}$ and for $e_1 = e,\ \alpha_1^* = \frac{1}{e}\frac{(n-1)\,r}{n^2}$.

P^2: $\alpha_2 = \frac{\partial}{\partial e_1} \frac{e_1^r}{\left((n-2)e^r + e_1^r\right)} \frac{(n-1)e^r}{\left((n-1)e^r + e_1^r\right)} = \frac{r(n-1)e^r e_1^{r-1}\left(e^{2r}(n-2)(n-1) - e_1^{2r}\right)}{\left((n-2)e^r + e_1^r\right)^2 \left((n-1)e^r + e_1^r\right)^2}$,

$$\underset{e_1=e}{\Leftrightarrow} \ \alpha_2^* = \frac{1}{e}\frac{(n^2 - 3n + 1)\,r}{(n-1)\,n^2}. \tag{6}$$

P^l: More generally, for prize $l \leq n$, we get

$$\alpha_l = \frac{\partial}{\partial e_1}\left(\tfrac{e_1^r}{(n-l)e^r+e_1^r}\tfrac{(n-l+1)e^r}{(n-l+1)e^r+e_1^r}\cdots\tfrac{(n-1)e^r}{(n-1)e^r+e_1^r}\right) =$$
$$\frac{\partial}{\partial e_1}\left(\tfrac{e_1^r}{(n-l)e^r+e_1^r}\prod_{x=1}^{l-1}\tfrac{(n-x)e^r}{(n-x)e^r+e_1^r}\right) = \frac{r\,(n-l)\,e^r e_1^{r-1}}{\left((n-l)\,e^r+e_1^r\right)^2}\prod_{x=1}^{l-1}\tfrac{(n-x)e^r}{(n-x)e^r+e_1^r}$$
$$-\sum_{x=1}^{l-1}\frac{e_1^r}{(n-l)e^r+e_1^r}\frac{r\,(n-x)\,e^r e_1^{r-1}}{\left((n-x)\,e^r+e_1^r\right)^2}\prod_{y=1,y\neq x}^{l-1}\frac{(n-y)\,e^r}{(n-y)e^r+e_1^r}. \tag{7}$$

Using the identity $\prod_{x=1}^{l-1}\frac{n-x}{n-x+1} = \frac{n-l+1}{n}$, we find the symmetric equilibrium expression as

$$\alpha_l^* = \frac{1}{e}\frac{r}{n}\left(\frac{n-l}{n-l+1} - \sum_{m=n-l+2}^{n}\frac{1}{m}\right) \tag{8}$$

which is decreasing in l for a given e, because the derivative of the first term in parentheses is negative and, for constant n and r, the sum is increasing for $l > 2$. Notice, moreover, that the last coefficient α_n must be negative since, for $l = n$, the above equals $-\frac{1}{e}\frac{r}{n}\sum_{m=2}^{n}\frac{1}{m}$.[8]

Hence the derivatives of the prize coefficients α are decreasing and $P^1 = P$ induces the contestant's highest equilibrium utilities. Since equilibrium efforts depend on the prize structure, however, it may be the case that equilibrium efforts are greater for some other prize structure (not maximizing the players' utilities). In order to show that this is not the case, we define effort independent coefficients $\beta_l = e\alpha_l$ for $l = 1, \ldots, n$ and obtain the first-order condition

$$c'(e)e = \sum_{l=1}^{n}\beta_l P^l. \tag{9}$$

Since we assume that $c'(e)e$ is increasing in e this condition ensures that when a SPSE exists it is unique. Taking equilibrium existence as given, we assume that the second-order condition holds locally at e^*, i.e.,

$$-\frac{1}{e^2}\sum_{l=1}^{n}\beta_l P^l - c''(e) < 0. \tag{10}$$

The solution e^* is then defined implicitly by $\frac{1}{e}\sum_{l=1}^{n}\beta_l P^l = c'(e)$ or $\frac{1}{e}q - c'(e) = 0$, where q is a constant parameter. Therefore

$$\frac{de^*}{dq} = -\frac{\dfrac{\partial\left(\frac{1}{e}q - c'(e)\right)}{\partial q}}{\dfrac{\partial\left(\frac{1}{e}q - c'(e)\right)}{\partial e}} = -\frac{\frac{1}{e}}{-\frac{1}{e^2}q - c''(e)} = \frac{e}{q + e^2c''(e)} \tag{11}$$

[8] Thus only monotonic prize structures $P^1 \geq P^2 \geq \cdots$ can be optimal. Moreover, assigning a positive prize to the contest loser reduces efforts. The same extends to all cases where (8) is negative. Approximately, for large n, $\alpha_l < 0$ when $\ln(n) - \ln(n-l+2) > \frac{n-l}{n-l+1}$ or roughly $l > \frac{2\mathrm{e}-n+\mathrm{e}n}{\mathrm{e}} \approx 0.63n$ (e is the exponential constant in this footnote).

and since we assume that the second-order condition holds at e^*, we know that $q+e^2c''(e)>0$ and $\frac{e}{q+e^2c''(e)}>0$. Thus we conclude that if e^* is a solution to player's i maximization problem (i.e., both first-order condition and second-order condition hold locally at e^*) then it is also true that e^* is monotonically increasing in q and an increase in $\sum \beta_l P^l$ increases e^*. □

Proof of proposition 2. We first prove that when condition (2) holds then there exists a SPSE under the winner take all prize structure. We show that if all players other than player $i \in \mathcal{N}$ exert effort e^*, then player i's best response is $e=e^*$. Player i's utility is then

$$U(e;e^*) = \frac{e^r}{e^r+(n-1)(e^*)^r}P - ae^b \text{ where } e^* = \left(\frac{1}{ab}\frac{(n-1)r}{n^2}P\right)^{\frac{1}{b}} \tag{12}$$

while, by playing e^*, the player gets $U(e^*;e^*)=\frac{1}{n}P-a\left(e^*\right)^b=\frac{1}{n}\left(1-\frac{1}{b}\frac{(n-1)r}{n}\right)P$. It is easy to show that both the first and second order conditions hold at e^* when (2) holds.[9] Assume that a critical point x exists where $\frac{d}{de}U(e;e^*)\Big|_{e=x}=0$.

a) We first show that if $0 \le x < e^*$, then $U(x;e^*) < U(e^*;e^*)$. Derive

$$\frac{d}{de}U(e;e^*) = \frac{(n-1)(e^*)^r re^{r-1}}{\left(e^r+(n-1)(e^*)^r\right)^2}P - abe^{b-1} \tag{13}$$

which equals at a critical point x

$$abx^{b-1} = \frac{(n-1)\left(e^*\right)^r rx^{r-1}}{\left(x^r+(n-1)\left(e^*\right)^r\right)^2}P. \tag{14}$$

Plugging the critical x^b from (14) into the player's objective (12), we obtain

$$U(x;e^*) = \frac{x^r}{x^r+(n-1)\left(e^*\right)^r}P - ax^b = \frac{x^r\left(bx^r+(n-1)\left(e^*\right)^r(b-r)\right)}{b\left(x^r+(n-1)\left(e^*\right)^r\right)^2}P. \tag{15}$$

Now, $U(x;e^*) < U(e^*;e^*)$ implies that

$$\frac{x^r\left(bx^r+(n-1)(e^*)^r(b-r)\right)}{b\left(x^r+(n-1)(e^*)^r\right)^2}P < \frac{1}{n}\left(1-\frac{1}{b}\frac{(n-1)\,r}{n}\right)P \tag{16}$$

[9] Deriving $U(e;e^*)$ with respect to e gives $\frac{d}{de}U(e;e^*)$ as

$$\frac{re^{r-1}\left(e^r+(n-1)\left(e^*\right)^r\right)-re^{2r-1}}{\left(e^r+(n-1)\left(e^*\right)^r\right)^2}P - abe^{b-1} = \frac{(n-1)\left(e^*\right)^r re^{r-1}}{\left(e^r+(n-1)\left(e^*\right)^r\right)^2}P - abe^{b-1}.$$

The second derivative gives $\frac{d^2}{de^2}U(e;e^*)|_{e=e^*}$ as

$$\frac{r\left(e^*\right)^{3r-2}(n-1)\left((r-1)(n-1)-(1+r)\right)}{n^3\left(e^*\right)^{3r}}P - ab\,(b-1)\left(e^*\right)^{b-2} = \frac{r\,(n-1)}{n^3\left(e^*\right)^2}P\left((n-2)\,r-nb\right)$$

and $(n-2)\,r-nb<0 \Leftrightarrow \frac{r}{b}<\frac{n}{n-2}$ which holds because we require $\frac{r}{b}\le\frac{n}{n-1}<\frac{n}{n-2}$.

which rearranges into

$$((e^*)^r - x^r)(n-1)(x^r(r+bn) + (e^*)^r(n-1)(bn-(n-1)r)) > 0 \tag{17}$$

which is true for $0 \le x < e^*$ precisely if (2) holds.

b) For $e > e^*$, we proceed to show that $U(e;e^*) - U(e^*;e^*) < 0$ or

$$\frac{e^r}{e^r + (n-1)(e^*)^r}P - ae^b - \frac{1}{n}\left(1 - \frac{1}{b}\frac{(n-1)r}{n}\right)P < 0. \tag{18}$$

Taking derivatives of $U(e;e^*) - U(e^*;e^*)$ with respect to e gives

$$\frac{e^r(e^*)^r(n-1)Pr - abe^b\left(e^r + (e^*)^r(n-1)\right)^2}{e\left(e^r + (e^*)^r(n-1)\right)^2} \tag{19}$$

which is negative—and hence there is no further critical point for $e > e^*$—if

$$\frac{r(n-1)}{abn^2}P < \frac{e^b(e^r + (e^*)^r(n-1))^2}{n^2 e^r (e^*)^r}. \tag{20}$$

Since the left-hand side equals $(e^*)^b$ this can be rearranged to $n\sqrt{(e^*)^{r+b}e^{r-b}} < e^r + (n-1)(e^*)^r$. Define $h(e) = n\sqrt{(e^*)^{r+b}e^{r-b}}$ and $g(e) = e^r + (n-1)(e^*)^r$—both strictly increasing functions in e. Notice that h and g intersect at $e = e^*$. Moreover, $\frac{d}{de}(g(e) - h(e))$ equals

$$re^{r-1} - \frac{1}{2}(r-b)n(e^*)^{\frac{1}{2}(r+b)}e^{\frac{1}{2}(r-b)-1} > 0 \Leftrightarrow \frac{1}{2r}(r-b)n < \left(\frac{e}{e^*}\right)^{\frac{1}{2}(b+r)}. \tag{21}$$

Since for $r \le \frac{n}{n-1}b$ the left-hand side is smaller than 1, this is true for all $e > e^*$, thus $g(e) - h(e) > 0$ and (18) holds for all $e > e^*$. As, given any prize structure, the symmetric equilibrium effort is unique, we also have the "only if" part. More precisely, if a symmetric equilibrium exists then $U(e^*;e^*) \ge U(0;e^*)$ only if $\frac{1}{n}\left(1 - \frac{1}{b}\frac{(n-1)r}{n}\right)P \ge 0 \;\Leftrightarrow\; r \le \frac{n}{n-1}b$. □

Proof of proposition 3. Assume a monotonic prize structure $P^1 \ge P^2 \ge \ldots \ge P^n \ge 0$, $\sum_l P^l = P$, for which a SPSE inducing positive efforts exists. We claim that if we change prizes to $\left(\frac{1}{n-1}(P - P^n), \ldots, \frac{1}{n-1}(P - P^n), P^n\right)$, then a SPSE that induces positive efforts still exists and equilibrium efforts decrease. Assuming that a SPSE exists, we first show that the efforts decrease:

$$\frac{1}{n-1}(P - P^n)\sum_{l=1}^{n-1}\beta_l + \beta_n P^n \le \sum_{l=1}^{n}\beta_l P^l. \tag{22}$$

This is true as we 'shift effort' from the first few prizes—with large weights β—to lower prizes. Formally, there exists an index s, $1 \le s < n-1$, such that $P^l \ge \frac{1}{n-1}(P - P^n)$ for any $l = 1, \ldots, s$

and $P^l < \frac{1}{n-1}(P-P^n)$ for $l = s+1, \dots, n-1$.[10] Now,

$$\frac{1}{n-1}(P-P^n)\sum_{l=1}^{n-1}\beta_l + \beta_n P^n \le \sum_{l=1}^{n}\beta_l P^l \Leftrightarrow \sum_{l=1}^{n-1}\beta_l\left(P^l - \frac{1}{n-1}(P-P^n)\right) \ge 0. \tag{23}$$

Since $\sum_{l=1}^{n-1}\beta_l\left(P^l - \frac{1}{n-1}(P-P^n)\right)$ equals

$$\sum_{l=1}^{s}\beta_l\left(P^l - \frac{1}{n-1}(P-P^n)\right) - \sum_{l=s+1}^{n-1}\beta_l\left(\frac{1}{n-1}(P-P^n) - P^l\right) \tag{24}$$

and

$$\begin{array}{c}\sum_{l=1}^{s}\beta_l\left(P^l - \frac{1}{n-1}(P-P^n)\right) \ge \beta_s\sum_{l=1}^{s}\left(P^l - \frac{1}{n-1}(P-P^n)\right) \text{ while} \\ \sum_{l=s+1}^{n-1}\beta_l\left(\frac{1}{n-1}(P-P^n) - P^l\right) \le \beta_s\sum_{l=s+1}^{n-1}\left(\frac{1}{n-1}(P-P^n) - P^l\right),\end{array} \tag{25}$$

we know that

$$\sum_{l=1}^{s}\left(P^l - \frac{1}{n-1}(P-P^n)\right) - \sum_{l=s+1}^{n-1}\left(\frac{1}{n-1}(P-P^n) - P^l\right) = 0 \tag{26}$$

and thus finally obtain that $\sum_{l=1}^{n-1}\beta_l\left(P^l - \frac{1}{n-1}(P-P^n)\right) \ge 0$. Since a SPSE exists for the original prize structure $U(e^*;e^*) = \frac{P}{n} - c(e^*) \ge P^n = U(0;e^*)$, we have the same inequality for the new prize structure (recall that $c(e)$ is monotonically increasing) which implies that for this new prize structure a SPSE inducing positive effort indeed exists. For this new prize structure—using the facts that $\sum \beta_l = 0$[11] and $P^n \le \frac{1}{n}P$—we have

$$\begin{aligned}
\frac{P}{n} - c(e^*) \ge P^n \quad &\Rightarrow \quad \frac{P}{n} - \frac{1}{b}\left(\frac{(P-P^n)}{n-1}\sum_{l=1}^{n-1}\beta_l + \beta_n P^n\right) \ge P^n \\
&\Rightarrow \quad \frac{1}{n}\left(1 - \frac{1}{b}\frac{n}{n-1}(-\beta_n)\right)P - \frac{1}{b}\frac{n}{n-1}\beta_n P^n \ge P^n \\
&\Rightarrow \quad \frac{1}{n}\left(1 + \frac{1}{b}\frac{n}{n-1}\beta_n\right)P \ge P^n\left(1 + \frac{1}{b}\frac{n}{n-1}\beta_n\right) \Rightarrow \left(1 + \frac{1}{b}\frac{n}{n-1}\beta_n\right) \ge 0 \\
&\Rightarrow \quad \frac{r}{b} \le \frac{n-1}{\sum_{m=2}^{n}\frac{1}{m}}.
\end{aligned}$$

Assume now that (3) holds. We show that we can find a prize structure 'close' to $(\frac{1}{n}P, \dots, \frac{1}{n}P)$ for which a SPSE inducing positive efforts exists. Recall that a given prize structure induces a unique

[10] If $s = n-1$, then this was already the original prize structure and we are done.

[11] Since $\sum_{l=1}^{n}\beta_l = \frac{(n-1)r}{n^2} + \sum_{l=2}^{n}\frac{r}{n}\left(\frac{n-l}{n-l+1} - \sum_{m=n-l+2}^{n}\frac{1}{m}\right) = \frac{r}{n}\left(\frac{n-1}{n} + \sum_{m=1}^{n-1}\frac{m-1}{m} - \sum_{m=2}^{n}\frac{m-1}{m}\right) = 0.$

symmetric equilibrium effort that solves (9), i.e.,

$$c'(e)e = \sum_{l=1}^{n} \beta_l P^l \Leftrightarrow e^* = \left(\frac{1}{ab}\sum_{l=1}^{n} \beta_l P^l\right)^{\frac{1}{b}} \Leftrightarrow c\left(e^*\right) = \frac{1}{b}\sum_{l=1}^{n} \beta_l P^l \tag{27}$$

where the coefficients β_l are functions of n and r (independent of e and P). Choose a small positive $\varepsilon \leq \frac{1}{n}P$ and consider the prize structure $(\frac{1}{n}P + \frac{1}{n-1}\varepsilon, \ldots, \frac{1}{n}P + \frac{1}{n-1}\varepsilon, \frac{1}{n}P - \varepsilon)$. If a SPSE exists under this prize structure, then it induces a positive effort of

$$e^* = \left(\frac{1}{ab}\left(\sum_{l=1}^{n-1} \beta_l \left(\frac{1}{n}P + \frac{1}{n-1}\varepsilon\right) + \beta_n \left(\frac{1}{n}P - \varepsilon\right)\right)\right)^{\frac{1}{b}}. \tag{28}$$

Since (3) holds, we indeed get that by exerting an effort of e^* the player achieves a higher utility than what she can achieve by exerting zero effort (while all other players exert $e^* > 0$), i.e.,

$$U(e^*; e^*) = \frac{P}{n} - c(e^*) \geq \frac{1}{n}P - \varepsilon = U(0; e^*). \tag{29}$$

This can be shown by expressing $c\left(e^*\right) = \frac{1}{b}\left(\sum_{l=1}^{n-1} \beta_l \left(\frac{1}{n}P + \frac{1}{n-1}\varepsilon\right) + \beta_n \left(\frac{1}{n}P - \varepsilon\right)\right)$ and again employing that $\sum \beta_l = 0$. Then (29) is equivalent to $\frac{1}{b}\left(\frac{1}{n-1}(-\beta_n) - \beta_n\right) \leq 1$ which, by substituting $\beta_n = -\frac{r}{n}\sum_{m=2}^{n} \frac{1}{m}$ gives $\frac{r}{b}\frac{1}{n-1}\sum_{m=2}^{n} \frac{1}{m} \leq 1$ which is true since (3) holds.
To ensure a global maximum we need to show that for every $e \notin \{0, e^*\}$, $U(e; e^*) < U(e^*; e^*)$ where

$$U(e^*; e^*) = \frac{P}{n} - c(e^*) = \frac{P}{n} - \frac{1}{b}\left(\sum_{l=1}^{n-1} \beta_l \left(\frac{1}{n}P + \frac{1}{n-1}\varepsilon\right) + \beta_n \left(\frac{1}{n}P - \varepsilon\right)\right) = \frac{P}{n} - \frac{r}{b}\frac{1}{n-1}\sum_{m=2}^{n} \frac{1}{m}\varepsilon.$$

a) We wish to show that for $e > e^*$, $\frac{d}{de}U(e; e^*) < 0$ implies

$$\frac{dU(e; e^*)}{de} = \left(\frac{n}{n-1}\varepsilon\right) \frac{(n-1)! r e^{r-1} \left(e^*\right)^{r(n-1)}}{\prod_{j=1}^{n-1}\left(e^r + (n-j)\left(e^*\right)^r\right)} \left(\sum_{l=1}^{n-1} \frac{1}{\left(e^r + (n-l)\left(e^*\right)^r\right)}\right) - abe^{b-1} < 0. \tag{30}$$

By rearranging and multiplying by $\sum_{m=2}^{n} \frac{1}{m}$ we get

$$\frac{r\varepsilon}{ab\left(n-1\right)} \sum_{m=2}^{n} \frac{1}{m} < \frac{e^{b-r} \prod_{j=1}^{n-1}\left(e^r + (n-j)\left(e^*\right)^r\right) \sum_{m=2}^{n} \frac{1}{m}}{n! \left(e^*\right)^{r(n-1)} \left(\sum_{l=1}^{n-1} \frac{1}{\left(e^r + (n-l)(e^*)^r\right)}\right)} \tag{31}$$

and using the fact that $(e^*)^b = \frac{r\varepsilon}{ab(n-1)}\sum_{m=2}^{n} \frac{1}{m}$ we obtain

$$\left(e^*\right)^{b+r(n-1)} < \frac{e^{b-r} \prod_{j=1}^{n-1}\left(e^r + (n-j)\left(e^*\right)^r\right) \sum_{m=2}^{n} \frac{1}{m}}{n! \left(\sum_{l=1}^{n-1} \frac{1}{\left(e^r + (n-l)(e^*)^r\right)}\right)}. \tag{32}$$

Now, for $e > e^*$

$$\frac{e^{b-r}\prod_{j=1}^{n-1}\left(e^r+(n-j)\left(e^*\right)^r\right)\sum_{m=2}^{n}\frac{1}{m}}{n!\left(\sum_{l=1}^{n-1}\frac{1}{(e^r+(n-l)(e^*)^r)}\right)} > e^{b-r}\left(e^*\right)^{rn} \tag{33}$$

and

$$(e^*)^{b+r(n-1)} < e^{b-r}(e^*)^{rn} \Leftrightarrow \left(\frac{e^*}{e}\right)^{b-r} < 1 \tag{34}$$

which is indeed true for $e > e^*$. **b**) For $e < e^*$, showing that $\frac{d}{de}U(e;e^*) > 0$ involves exactly the same steps (30)–(34) as under a) for the reversed inequality. □

<u>*Proof of proposition 4.*</u> We define k as the smallest integer such that the prize structure (4) induces a symmetric equilibrium effort (27) which is smaller or equal to the optimal effort $c^{-1}\left(\frac{P}{n}\right)$. Thus k is the smallest integer such that $\frac{1}{k}\sum_{l=1}^{k}\beta_l \leq \frac{b}{n}$. We know that such an integer exists since the left-hand side is decreasing in k (for $k < n$)

$$\frac{1}{k}\sum_{l=1}^{k}\beta_l \geq \frac{1}{k+1}\sum_{l=1}^{k+1}\beta_l \quad \Leftrightarrow \quad \frac{1}{k}\sum_{l=1}^{k}\beta_l \geq \beta_{k+1} \tag{35}$$

and since β_l is decreasing with l we have $\frac{1}{k}\sum_l \beta_l > \frac{1}{k}\sum_l \beta_k = \beta_k > \beta_{k+1}$ and we establish (35). Moreover, for $k = 1$ we have $\beta_1 = \frac{(n-1)r}{n^2} > \frac{b}{n}$ since (2) is violated, and since (3) holds, we know that, for $k = n-1$, it is true that $\frac{1}{k}\sum_{l=1}^{k}\beta_l \leq \frac{b}{n}$. We thus need to show that given k we can find an $0 \leq \varepsilon < \frac{1}{k}P$ such that the prize structure (4) induces the optimal effort, i.e.,

$$e^*(k) = \left(\frac{1}{ab}\left(\sum_{l=1}^{k-1}\beta_l\left(\frac{P}{k}+\frac{1}{k-1}\varepsilon\right)+\beta_k\left(\frac{1}{k}P-\varepsilon\right)\right)\right)^{\frac{1}{b}} = \left(\frac{1}{a}\frac{P}{n}\right)^{\frac{1}{b}}. \tag{36}$$

We find ε by ensuring that the utility of the players is minimized and equal to zero, i.e.,

$$(k-1)\frac{1}{n}\left(\frac{P}{k}+\frac{1}{k-1}\varepsilon\right)+\frac{1}{n}\left(\frac{P}{k}-\varepsilon\right)-\frac{1}{b}\left(\sum_{l=1}^{k-1}\beta_l\left(\frac{P}{k}+\frac{1}{k-1}\varepsilon\right)+\beta_k\left(\frac{1}{k}P-\varepsilon\right)\right)=0$$
$$\frac{1}{b}\left(\frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l-\beta_k\right)\varepsilon = \left(\frac{1}{n}-\frac{1}{bk}\sum_{l=1}^{k}\beta_l\right)P.$$

We also know that $\frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l > \frac{b}{n}$ and that $\frac{1}{k}\sum_{l=1}^{k}\beta_l = \frac{1}{k}\sum_{l=1}^{k-1}\beta_l + \frac{1}{k}\beta_k \leq \frac{b}{n}$. Thus

$$\left(\frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l-\beta_k\right) \geq k\left(\frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l-\frac{b}{n}\right) > 0. \tag{37}$$

Therefore we have

$$\varepsilon = \frac{\left(\frac{b}{n}-\frac{1}{k}\sum_{l=1}^{k}\beta_l\right)}{\left(\frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l-\beta_k\right)}P > 0. \tag{38}$$

Finally we need to show that $\varepsilon < \frac{1}{k}P$ implying that

$$\frac{b}{n} - \frac{1}{k}\sum_{l=1}^{k}\beta_l \;<\; \frac{1}{k}\left(\frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l - \beta_k\right) \Leftrightarrow \frac{b}{n} < \frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l \tag{39}$$

which follows from the definition of k. Using this in (36), we obtain efforts $e^*(k)$ as

$$\left(\frac{1}{ab}\left(\sum_{l=1}^{k-1}\beta_l\left(\frac{1}{k}+\frac{1}{k-1}\frac{\left(\frac{b}{n}-\frac{1}{k}\sum_{l=1}^{k}\beta_l\right)}{\left(\frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l-\beta_k\right)}\right)+\beta_k\left(\frac{1}{k}-\frac{\left(\frac{b}{n}-\frac{1}{k}\sum_{l=1}^{k}\beta_l\right)}{\left(\frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l-\beta_k\right)}\right)\right)P\right)^{\frac{1}{b}}$$
$$= \left(\frac{1}{ab}\frac{b}{n}\left(\frac{\frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l-\beta_k}{\frac{1}{k-1}\sum_{l=1}^{k-1}\beta_l-\beta_k}\right)P\right)^{\frac{1}{b}} = \left(\frac{1}{a}\frac{P}{n}\right)^{\frac{1}{b}}. \qquad \square$$

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