Sequential bargaining with common values: The case of bilateral incomplete information*

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Abstract

We study the alternating-offers bargaining problem of assigning an indivisible and commonly valued object to one of two players in return for some payment to the other. The players are asymmetrically and imperfectly informed about the object's value and have veto power over any settlement. There is no depreciation during the bargaining process which involves signalling of private information on both sides. We characterise the perfect Bayesian equilibrium of this game which has a unique outcome under strictly increasing offers. Equilibrium agreement is reached gradually and non-deterministically. The player obtaining the more precise signal appropriates a rent. (JEL *C73, C78, D44, D82, J12.* Keywords: *Sequential bargaining, Common values, Incomplete information, Repeated games, cake cutting.*)

1 Introduction

This note extends the results of the sequential bargaining model with one-sided incomplete information introduced in Schweinzer (2010) to the case of incomplete information on both sides. We answer the question of what, specifically, the value of private information is in such a common value bargaining environment when players have differential signal accuracy on the commonly valued object. The problem is studied in an alternating-offers bargaining model where two privately informed players reveal their private information through the offers they make to each other. Our setup features an indivisible object that is of either high or low value. Both players know these possible values and initially receive signals with publicly known differential precision informing them privately and imperfectly on these realisation of the value. Players are risk-neutral, infinitely patient and possess similar bargaining power as offers are made alternatingly. In the unique equilibrium outcome, the more precisely informed player obtains an information rent while the less precisely informed player finds it interim individually

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rational to participate. A multitude of applications of this ‘partnership-dissolution’ model are discussed in Schweinzer (2010) who also provides an overview of the applicable literature.

The equilibrium dynamics under bilateral incomplete information are a natural extension of the unilateral case: In the essentially unique perfect Bayesian equilibrium parameterised by the players’ signal-accuracies, high signal players keep increasing their offers by the minimal unit at a time up to approximately half the high-state expectation of the object and then accept the opponent’s offer. Low signal players similarly keep increasing their offers with probability 1 up to a certain threshold which depends on priors and signal accuracies. After reaching this bidding threshold, they mix between minimally increasing their own offers and accepting the opponent’s last offer until they reach the stage where they follow their high-type counterparts in accepting. Depending on the individual signal accuracies, however, this basic strategy may take different forms and give rise to other equilibria of the bargaining game. Disregarding knife-edge cases, though, these are all (essentially) unique for their parameters. We provide an example in the appendix which illustrates this for *any* combination of signal accuracies. We focus on this particular pair of signal intervals because it is hardest to implement. Once this equilibrium is characterised, the (likewise essentially unique) equilibria for different signal precisions are straightforward to find by relaxing some of the mixture or continuation requirements at the final stages of the game.

Our model of incomplete information on both sides encompasses all situations where each of the asymmetrically informed players can form a more precise conditional expectation on the object’s value if they were to learn the opponent’s signal than without that knowledge. This note concentrates on the differences to the case of incomplete information on one side. In the interest of brevity, we employ existing results from the one-dimensional case whenever possible.

Related literature

The dynamic bargaining game we study is a repeated game of incomplete information. First results on such games were obtained by Aumann and Maschler (1966); the most notable generalisations are by Mertens, Sorin, and Zamir (1994). The typical approach there is to derive average payoffs from long-run interactions which do not arise naturally in our context. There is a rich literature on bargaining with incomplete information. Extensive surveys on bargaining under incomplete information are presented, for instance, by Ausubel, Cramton, and Deneckere (2002) but, to present, the only analysis of incomplete information bargaining over an object’s pure common value allowing for bids by both players is Schweinzer (2010). As mentioned above, that paper also summarises the applicable literature. This paper provides the extension to asymmetric incomplete information on both sides by introducing imperfectly informative signals to *both* players. In contrast to the one sided incomplete information scenario, the better informed player need not necessarily know everything the less-well informed player knows but gains additional information by learning the lower-precision signal. We model this signal accuracy as a discrete version of ideas developed by Athey and Levin (1998) and Persico (2000) as an extension of the concept of signal sufficiency.

2 The model

We consider two identical, risk-neutral players P1, P2 and two possible common values for an indivisible object $\theta \in \{\underline{\theta}, \bar{\theta}\}$, $\bar{\theta} \in \mathbb{R}$. We normalise $\underline{\theta} = 0$ and assume $\bar{\theta} \geq 3$ to avoid trivialities. Nature chooses θ with the publicly known probability $\varphi^0 = \mathrm{pr}(\theta = \bar{\theta}) = {}^1/_2$, although there is nothing special about this number. Subsequently, Nature sends a private signal $s_i \in \{\underline{s}, \overline{s}\}$ to each player $i = 1, 2$; $\mathbf{s}$ denotes the pair of signals s_1, s_2. Signals have publicly known accuracy $p_i = \mathrm{pr}(s_i = \bar{s}_i|\bar{\theta}) = \mathrm{pr}(s_i = \underline{s}_i|\underline{\theta}) \in [{}^1/_2, 1]$ where $p = {}^1/_2$ is uninformative and $p = 1$ is fully revealing.[1] We assume p_1, p_2 to be iid. conditional on the realised value θ and write $\mathbf{p} = p_1, p_2$. Thus, the unconditional ex-ante probability of receiving a high signal is $\mathrm{pr}(s_i = \bar{s}) = \varphi^0 = {}^1/_2$ for both players. The more accurately informed player, ie. the player with the higher p, is called P1. We define Pi's beliefs as the conditional probability with which he believes the other player to have received a high signal, ie. $\varphi_i = \mathrm{pr}(s_j = \bar{s}|s_i)$ where we typically index players by $i \in \{1, 2\}$, $j = 3 - i$. On the basis of his own signal, player Pi updates these beliefs through his observation of his opponent's bidding behaviour throughout the game.

The game starts with the better informed P1 offering a payment o_1^1 (subscripts are players, superscripts time periods) to P2 for sole ownership of the object.[2] Pure offers o_i^t, $t > 0$ are restricted to the set of admissible bids $B = \{0, 1, \dots, \bar{B}\} \subset \mathbb{N}$ where $\bar{B} > \bar{\theta}$ ('all the money in the world'). This defines the minimal offer increment as 1 (currency unit). The terms offers and bids are used interchangeably. If Pj accepts Pi's offer, Pi pays the offered amount to Pj, Pi gets the object and the game is over. If Pj does not accept Pi's offer, nothing is paid, and Pj makes an own offer. Players go on making alternating offers until one player accepts an offer by accepting. This must happen at the latest when the highest possible bid $\overline{B}$ is reached. Adding the option for players to terminate negotiations with zero payoffs at each stage changes neither the analysis nor the results (as this option is dominated by accepting the previous offer).

We set $o_2^0 = o_1^{-1} = 0$ equal to the low value of the object and require offers to be strictly increasing over time, ie. all continuation *increments* over the last own offer $o_i^t - o_i^{t-2} > 0$.[3] Conversely, $o_i^t - o_i^{t-2} \leq 0$ is interpreted as accepting and denoted 'q'. Mixed offers attach probability α_i^t to the pure continuation bid o_i^t and the complementary probability to accepting. We denote such mixed actions by $\beta_i^t = [\alpha_i^t : o_i^t, q]$.[4]

Pi's strategy β_i consists of the sequence of potentially mixed stage actions for each possible plan of the opponent. Players observe the opponents' realised offers and enjoy perfect recall. The players' final expected payoffs are written $u_i(\beta|s)$ and consist of the object's value minus

[1] The case of $\mathbf{p} = (1, {}^1/_2)$ is the game with incomplete information on one side analysed in Schweinzer (2010).

[2] This is not without loss of generality. However, the influence on the equilibrium of having the better informed P1 start the game is minimal when θ is large.

[3] A generalisation to arbitrary bids introduces an infinite game and can be accomplished as in Schweinzer (2010). That is, one can show that a modified version of the equilibrium of the game with strictly increasing offers remains an equilibrium of the extended game allowing for fully general offers, ie. dropping the requirement for bids to be strictly increasing. As the involved techniques are not different from those employed in Schweinzer (2010) we do not duplicate that analysis.

[4] We refrain from a more general definition of a mixed stage action (over a larger support of pure actions) because lemma 6 shows that we need nothing more complicated than the above.

payment made for the winner of the object and the payment received for the loser. Pi's payoff when accepting an offer at t is written as $u_i^t(q)$.

For ease of exposition, we introduce some further notation. $\overline{\mathrm{P}i}$ and $\underline{\mathrm{P}i}$ are the high-type and low-type of Pi, respectively. Since $\bar{s}_i$ is player i's high signal and $\underline{s}_i$ his low signal, we write the possible signal profiles as $\underline{\underline{s}}$, $\bar{\underline{s}}_1^2$, $\bar{\underline{s}}_2^1$, and $\bar{\bar{s}}$. Similarly, we write the object's expected value given the possible signal combinations as $\underline{\underline{\mathrm{E}}}$, $\bar{\underline{\mathrm{E}}}_1^2$, $\bar{\underline{\mathrm{E}}}_2^1$, $\bar{\bar{\mathrm{E}}}$. The belief-dependent expectations based on one player's signal plus his beliefs on the opponent's signal are written $\underline{\mathrm{E}}_1$, $\bar{\mathrm{E}}^1$, $\underline{\mathrm{E}}_2$, $\bar{\mathrm{E}}^2$. Finally, we denote the low-type Pi's, time-t belief by $\underline{\varphi}_i^t$ and the high type's belief by $\bar{\varphi}_i^t$.

We use $\mathcal{Q}_B$ to denote this alternating-offer bargaining game with asymmetric incomplete information over common values on both sides and no discounting. We now state the definitions required to formulate our results in section 4 which contains all proofs and details. An extensive example is provided in the appendix.

3 Discussion

For any chosen value $\bar{\theta}$, there are many different equilibria for the possible signal precisions $\mathbf{p}$. A full characterisation of all these equilibria would be impossible and—as many equilibria share their determining features—also unnecessary for understanding the game's underlying dynamics. For the latter purpose it suffices to investigate the most demanding class of parameterised equilibria in terms of information revelation. This class allows for making use of results known from the case of incomplete information on only one side Schweinzer (2010) for which the following assumption holds trivially. Thus the class of equilibria investigated is nonempty. The equilibria for all other parameter regions can be obtained by relaxing some of the involved conditions.

Assumption (A). *We restrict attention to signal accuracies p_1, p_2 where, i) a high-type player makes the last equilibrium continuation bid of the game at some period t_f wp1, ii) the low type moving at this period accepts wp1, iii) the low type moving at $t_f - 1$ mixes between continuing and accepting, and iv) both types of the player moving at $t_f + 1$ accept wp1.*[5] *Moreover, we assume that the influence of the bidding grid is low in the sense that $\lfloor \bar{\bar{E}} \rfloor > \bar{\underline{E}}_2^1$.*

We stress that this is an assumption on the set of signal parameters, not on the players' behaviour. We call the set of signal precisions satisfying this assumption $\tilde{\mathbf{p}}$ and the correspondingly parameterised equilibrium candidate $\beta^*(\tilde{\mathbf{p}})$. Under this candidate, both high-type players always increase their offers minimally wp1 until the sum of the most recent offers reaches the object's high-signal expectation while both low-type players always mix between accepting and increasing their offer minimally, starting from some period t_s and accepting one period before the high types accept.[6] Both low-type players mix in equilibrium in order to make their low-type opponents mix in turn—thus they start mixing at the same period ± 1. We call the part of

[5] The precise requirements this implies for signal accuracies and priors are detailed in (17) and (18).

[6] We use the abbreviations wp1 and wpp for 'with probability one' and 'with positive probability,' respectively.

the game where both low types mix the 'main' game. Compared to the game with incomplete information on one side, this creates the complication that these mixtures change both players' beliefs which are crucial for the calculation of the continuation payoff expectations (and thus the own next-period mixing condition). Hence the low-type's mixture conditions at each stage of the main game are harder to enforce than in the unilateral incomplete information case.

It is helpful for understanding the equilibrium dynamics to see that, on the one hand, $\underline{\text{P2}}$ is made indifferent between accepting and minimally increasing through her beliefs $\underline{\varphi}_2^t$ set by $\underline{\text{P1}}$'s previous period continuation probability α_1^{t-1}. $\underline{\text{P1}}$, on the other hand, is made indifferent between accepting and minimally increasing through $\underline{\text{P2}}$'s next period's continuation probability α_2^{t+1}. Thus the mixing dynamics of the game are, for odd $\bar{\bar{\text{E}}}$, such that $\underline{\text{P1}}$ starts mixing as soon as $\underline{\text{P2}}$'s continuation payoffs from her prior-based $\beta^*(\tilde{\mathbf{p}})$ are below her acceptance payoffs. $\underline{\text{P1}}$'s mixing with *any* $\alpha_1^{t_s}$ (based on his prior beliefs $\underline{\varphi}_1^{t_s}$) determines a unique mixture probability $\alpha_2^{t_s+1}$ at the subsequent stage. This mixture probability, in turn, determines a unique belief $\underline{\varphi}_1^{t_s+2}$ which allows $\underline{\text{P1}}$'s mixing and thus in turn determines a unique $\alpha_2^{t_s+3}$ and so on until the final period $t_f - 1$ where $\underline{\text{P2}}$ mixes. We refer to this chain of reasoning which determines a unique α_2^t at all even t in the main game as the *forward chain*.

Conversely, $\underline{\text{P2}}$'s final belief $\underline{\varphi}_2^{t_f-1}$ (11)—before $\underline{\text{P1}}$ accepts at t_f—is determined from her mixture condition over terminal payoffs. This indifference belief, in turn, determines a unique $\alpha_1^{t_f-2}$ which generates this belief. This $\alpha_1^{t_f-2}$, however, also determines $\underline{\text{P2}}$'s payoffs at t_f-3 and thus requires a unique $\underline{\varphi}_2^{t_f-3}$ in order to ensure $\underline{\text{P2}}$'s indifference. This belief again determines uniquely its generating $\alpha_1^{t_f-4}$ and so on until $\underline{\text{P1}}$'s first mixture period is reached at t_s. Thus all α_1^t for odd t are uniquely determined through this *backward chain* from $\underline{\text{P2}}$'s terminal equilibrium belief (11). In order to determine the equilibrium periods t_s and t_f referred to above as well as all equilibrium actions and beliefs, we require both the forward and backward chains.[7]

The dynamics for even $\bar{\bar{\text{E}}}$ are similar: $\underline{\text{P2}}$ accepts at the continuation bid $\bar{\bar{\text{E}}}$ which determines $\underline{\text{P1}}$'s previous period's beliefs. These, in turn, determine $\underline{\text{P2}}$'s mixture one more period ahead and so on until the prior-based equilibrium continuation payoff exceeds the acceptance payoff and $\underline{\text{P2}}$ stops mixing but continues minimally wp1. Since this leaves $\underline{\text{P1}}$'s terminal mixture probability undefined, it determines $\underline{\text{P2}}$'s mixture probability one period *backwards*. Hence there is no need for $\underline{\text{P1}}$ to mix *before* $\underline{\text{P2}}$'s first mixture because his following period mixture directly manipulates her continuation payoff.

The basic requirement from $\beta^*(\tilde{\mathbf{p}})$ is that, at each stage of the main game, a low-type player must be indifferent between all pure actions contained in the support of his mixed action.[8] Thus $\underline{\text{P}i}$ mixes at t if $u_i^t(q) = u_i^{t+1}(\beta^*|s)$ or

$$o_j^{t-1} = (1 - \underline{\varphi}_i^t)[(1 - \alpha_j^{t+1})(\underline{\underline{\text{E}}} - o_i^t) + \alpha_j^{t+1} o_j^{t+1}] + \underline{\varphi}_i^t o_j^{t+1}$$

[7] This is intuitive as 'life can only be understood going backwards, but it must be lived going forwards.' (Søren Kirkegaard)

[8] In the following discussion, since we do not discuss other parameter regions than $\tilde{\mathbf{p}}$, we omit the parameter specification on the equilibrium (candidate) $\beta^*(\tilde{\mathbf{p}})$ if there is no risk of confusion.

resulting in j's mixture probability

$$ {}^{*}\alpha_j^{t+1} = \frac{(1-\underline{\varphi}_i^t)(o_i^t + o_j^{t+1} - \underline{\underline{\mathrm{E}}}) - (o_j^{t+1} - o_j^{t-1})}{(1-\underline{\varphi}_i^t)(o_i^t + o_j^{t+1} - \underline{\underline{\mathrm{E}}})} = \frac{(1-\underline{\varphi}_i^t)(t - \underline{\underline{\mathrm{E}}} + 1) - 1}{(1-\underline{\varphi}_i^t)(t - \underline{\underline{\mathrm{E}}} + 1)}. \tag{1} $$

where the rhs equality holds only on the equilibrium path. In addition, $\underline{\mathrm{P}i}$'s beliefs $\underline{\varphi}_i^t$ must stem from the application of Bayes' rule

$$ \begin{aligned} \underline{\varphi}_i^t = \mathrm{pr}(\bar{s}_j|\underline{s}_i) &= \frac{\mathrm{pr}(o_j^{t-1}|\bar{s}_j)\,\mathrm{pr}(\bar{s}_j)}{\mathrm{pr}(o_j^{t-1}|\bar{s}_j)\,\mathrm{pr}(\bar{s}_j) + \mathrm{pr}(o_j^{t-1}|\underline{s}_j)\,\mathrm{pr}(\underline{s}_j)} = \frac{\underline{\varphi}_i^{t-2}}{\underline{\varphi}_i^{t-2} + (1-\underline{\varphi}_i^{t-2})\alpha_j^{t-1}}, \\ \bar{\varphi}_i^t = \mathrm{pr}(\bar{s}_j|\bar{s}_i) &= \frac{\bar{\varphi}_i^{t-2}}{\bar{\varphi}_i^{t-2} + (1-\bar{\varphi}_i^{t-2})\alpha_j^{t-1}}. \end{aligned} \tag{2} $$

Depending on the object's expectations, β^* defines the terminal beliefs (11) and (12) through the low types' final equilibrium mixing condition. We define the type and history dependent equilibrium strategy profile $\beta^* = (\beta_1^*, \beta_2^*)$, $o_i^t \in B$, as

$$ \begin{aligned} \beta_i^*(\underline{s}) &= \left\{ \begin{array}{rl} o_i^t = \text{`q'}: & \text{if } o_i^{t-2} + o_j^{t-1} \geq \bar{\mathrm{E}}^i - 1, \\ \beta_i^t = \left[\alpha_i^t : o_i^t = o_i^{t-2}+1, q\right]: & \text{if } u_j^{t+1}(q) > \underline{\varphi}_j^1 u_j^{t+1}(\beta^*|\underline{s}_j^i) + \\ & \qquad (1-\underline{\varphi}_j^1) u_j^t(\beta^*|\underline{s}), \\ o_i^t = o_i^{t-2}+1: & \text{otherwise}, \end{array} \right\}_{\forall t>0} \\ \beta_i^*(\bar{s}) &= \left\{ \begin{array}{rl} o_i^t = \text{`q'}: & \text{if } o_i^{t-2} + o_j^{t-1} \geq \bar{\mathrm{E}}^i - 1, \\ o_i^t = o_i^{t-2}+1: & \text{otherwise} \end{array} \right\}_{\forall t>0} \end{aligned} \tag{3} $$

where $u_i^t(\beta|s)$ are i's continuation payoffs following β^* from t onwards. Notice that, in equilibrium, both players' beliefs are uniquely defined by (2) at each information set. Their beliefs are not fully defined, however, when they accept wp1 for *any* belief. This situation cannot occur in equilibrium: Pi only accepts wp1 following a fully separating action by $\underline{\mathrm{P}j}$ which implies that $\varphi_i = 1$ if a continuation is observed. Off-equilibrium-path, however, this situation cannot be ruled out and we set $\varphi_i = 1$ at all information sets where any belief leads to accepting, ie. where Pi's beliefs do not matter.

An equilibrium argument removes a related problem in the game before the low types start to mix. If Pi observes an off-equilibrium-path bid, her beliefs cannot be deduced from Bayes' rule. But as long as Pi's continuation payoff is larger than her current acceptance payoff for *any* belief φ_i, then mixing by $\underline{\mathrm{P}j}$ (over any set of pure actions) cannot be an equilibrium action since it is not followed by mixing of $\underline{\mathrm{P}i}$. Hence the only *equilibrium* response consistent with $\underline{\mathrm{P}i}$ not mixing is that neither of Pj's types mixes. But then Pi does not update her belief and $\varphi_i = \varphi_i^1$ throughout that initial phase. Thus both players' *equilibrium* beliefs are pinned down uniquely in the initial phase.

4 Results

Our main argument is an induction proof from the highest possible equilibrium bid back to the first period of the game. It relies on assumption (A) and its first part consists of lemma 2 showing that no bid o_i^t in excess of $\bar{\bar{\mathrm{E}}} - o_j^{t-1}$ can be part of any possible equilibrium of the game. We then show that the high types will not accept wpp before this offer and call the equilibrium period of the highest possible offer t_f. We go on to show that this period is reached by all types wpp in lemma 4 and reason that no separating action can exist apart from that at t_f. Lemma 6 makes our argument against jump bids (increasing the offer by more than the minimal increment) which allows to uniquely pin down equilibrium mixtures and beliefs in lemma 7. Proposition 1 summarises the above arguments to conclude that β^* is indeed an equilibrium of $\mathcal{Q}_B$ for the assumed parameters. Finally, proposition 2 finds the period t_s where the first low-type player mixes between accepting and continuing given the highest high-type equilibrium offer at t_f. The pair of these two equilibrium periods (t_s, t_f) are endogenised in the equilibrium β^*. We start with an accounting argument used repeatedly in the successive reasoning.

Lemma 1. *For $p_1 > p_2$, $p_1, p_2 \in [^1/_2, 1]$,*

$$0 < \underline{\underline{E}} < \underline{E}_1 < \underline{E}_2 < \bar{\underline{E}}_1^2 < \bar{\underline{E}}_2^1 < \bar{E}^2 < \bar{E}^1 < \bar{\bar{E}} < \bar{\theta}. \tag{4}$$

Proof. Players update initial beliefs $\varphi_i = \mathrm{pr}(s_j = \bar{s}|s_i)$ after observing their signals to

$$\underline{\varphi}_1^1 = p_1 + p_2 - 2p_1p_2, \quad \underline{\varphi}_2^1 = (1-p_1)p_2 + p_1(1-p_2),$$
$$\bar{\varphi}_1^1 = p_1p_2 + (1-p_1)(1-p_2), \quad \bar{\varphi}_2^1 = (1-p_1)(1-p_2) + p_1p_2.$$

Then the claim (4) follows from the definitions of the object's expectations

$$\left.\begin{array}{l} \bar{\underline{\mathrm{E}}}_1^2 = \dfrac{\bar{\theta}(1-p_1)p_2}{p_1+p_2-2p_1p_2}, \ \underline{\underline{\mathrm{E}}} = \dfrac{\bar{\theta}(1-p_1)(1-p_2)}{(1-p_1)(1-p_2)+p_1p_2)} \\ \bar{\bar{\mathrm{E}}} = \dfrac{\bar{\theta}p_1p_2}{p_1p_2+(1-p_1)(1-p_2)}, \ \bar{\underline{\mathrm{E}}}_2^1 = \dfrac{\bar{\theta}p_1(1-p_2)}{p_1+p_2-2p_1p_2} \end{array}\right\} \begin{array}{l} \underline{\mathrm{E}}_2 = (1-\underline{\varphi}_2^1)\underline{\underline{\mathrm{E}}} + \underline{\varphi}_2^1\bar{\underline{\mathrm{E}}}_2^1 = (1-p_2)\bar{\theta}, \\ \underline{\mathrm{E}}_1 = (1-\underline{\varphi}_1^1)\underline{\underline{\mathrm{E}}} + \underline{\varphi}_1^1\bar{\underline{\mathrm{E}}}_1^2 = (1-p_1)\bar{\theta} \end{array}$$

and $\bar{\mathrm{E}}^1$, $\bar{\mathrm{E}}^2$ are defined accordingly. Since $p_1p_2 > (1-p_1)(1-p_2)$ we know that $\underline{\underline{\mathrm{E}}} < \bar{\bar{\mathrm{E}}}$ and since $p_1 > p_2$, $\bar{\underline{\mathrm{E}}}_2^1 > \bar{\underline{\mathrm{E}}}_1^2$. Similarly $\bar{\underline{\mathrm{E}}}_1^2 > \underline{\underline{\mathrm{E}}}$ because, if not, $2p_2 < \frac{p_1}{1+p_1}$ which is a contradiction, and $\bar{\bar{\mathrm{E}}} > \bar{\underline{\mathrm{E}}}_2^1$ because, if not, $2p_1 < \frac{p_2}{1+p_2}$ which is again a contradiction. $\bar{\mathrm{E}}^1 > \bar{\mathrm{E}}^2$ and $\underline{\mathrm{E}}_1 < \underline{\mathrm{E}}_2$ because $\underline{\varphi}_i^0 < \bar{\varphi}_i^0$. Finally, $\underline{\underline{\mathrm{E}}} < \underline{\mathrm{E}}_i$ because $\underline{\mathrm{E}}_i$ is a convex combination between $\underline{\underline{\mathrm{E}}}$ and something bigger and, likewise, $\bar{\bar{\mathrm{E}}} > \bar{\mathrm{E}}^i$. □

The following lemma determines the highest possible continuation bids in any equilibrium of the game. Figure 1 shows these last equilibrium stages for both even and odd final periods.

Lemma 2. *In any equilibrium satisfying (A), i) $\overline{Pi}$ accepts wp1 at period $\hat{t}$, if*

$$o_i^{\hat{t}-2} + o_j^{\hat{t}-1} > \bar{\bar{E}} - 1 \tag{5}$$

and ii) $\underline{Pi}$ *quits wp1 at period* $\hat{t}$*, if*

$$o_{-i}^{\hat{t}-1} + o_i^{\hat{t}} \geq \bar{\bar{E}} - 1. \tag{6}$$

Proof. As we are interested in the highest possible bids, we only need to consider beliefs are $\bar{\varphi}_i = 1$. Then time-t acceptance payoffs are

$$\begin{aligned} u^t(q) = &\ (o_2^{t-1}, \bar{\bar{\mathrm{E}}} - o_2^{t-1}) \text{ if P1 accepts at odd } t, \\ u^t(q) = &\ (\bar{\bar{\mathrm{E}}} - o_1^{t-1}, o_1^{t-1}) \text{ if P2 accepts at even } t. \end{aligned} \tag{7}$$

If no player accepts before, one bidder must eventually bid the highest possible bid $\bar{B}$. Suppose that Pi makes this last admissible continuation bid $o_i^{\hat{t}} = \bar{B} > \bar{\theta} \geq \bar{\bar{\mathrm{E}}}$ at some period $\hat{t}$. Then, at $\hat{t}+1$, Pj must accept Pi's offer through accepting wp1. Payoffs at $\hat{t}+1$ are then $u_i = \bar{\bar{\mathrm{E}}} - o_i^{\hat{t}}$ and $u_j = o_i^{\hat{t}}$. Since

$$u_i = \bar{\bar{\mathrm{E}}} - o_i^{\hat{t}} = \bar{\bar{\mathrm{E}}} - \bar{B} < 0,$$

however, Pi can do better by accepting at $\hat{t}$ and accepting Pj's offer $o_j^{\hat{t}-1} > 0$. Knowing that Pj will also accept if her time $t-1$ acceptance payoff exceeds her time t continuation payoff, we obtain Pi's accepting condition at $\hat{t}$ wp1 as

$$\bar{\bar{\mathrm{E}}} - o_i^{\hat{t}} < o_j^{\hat{t}-1} \text{ or } o_j^{\hat{t}-1} + o_i^{\hat{t}} > \bar{\bar{\mathrm{E}}}. \tag{8}$$

Since the minimal admissible increment for continuation is 1, this corresponds to i's accepting condition at $\hat{t}$ wp1 if

$$o_j^{\hat{t}-1} + o_i^{\hat{t}-2} > \bar{\bar{\mathrm{E}}} - 1. \tag{9}$$

Thus no time-t equilibrium continuation bid by player i can be higher than $\bar{\bar{\mathrm{E}}} - o_j^{t-1}$. As we choose to analyse the most demanding equilibrium where high type players continue up to exactly this value, assumption (A) restricts attention to equilibria where $\overline{Pi}$'s highest equilibrium bid is exactly $\lfloor \bar{\bar{\mathrm{E}}} \rfloor - o_j^{t-1}$. Since this is the highest possible bid, Pj necessarily accepts the following period.

In the low-signal branches of the game, $\underline{Pi}$ will only continue wpp if Pj continues wpp at the following stage. If not, she will accept wp1 the period before. Thus $\underline{Pi}$ will at the latest accept wp1 in equilibrium at $\hat{t}$ if

$$o_j^{\hat{t}-1} + o_i^{\hat{t}} \geq \bar{\bar{\mathrm{E}}} - 1 \text{ or } o_j^{\hat{t}-1} + o_i^{\hat{t}-2} \geq \bar{\bar{\mathrm{E}}} - 2. \qquad \square$$

Thus there is an equilibrium period t_f after which high-type players accept for certain. Since low types have necessarily lower expectations than high types, they will certainly accept at that period, too. Assumption (A) allows us to focus on parameters for which $\overline{Pi}$ continues wp1 at t_f and $\underline{Pi}$ accepts wp1 at $t_f - 1$. This is the only separating action of the game.

Lemma 3. *The only separating equilibrium satisfying assumption (A) occurs at a period* t_f*,*

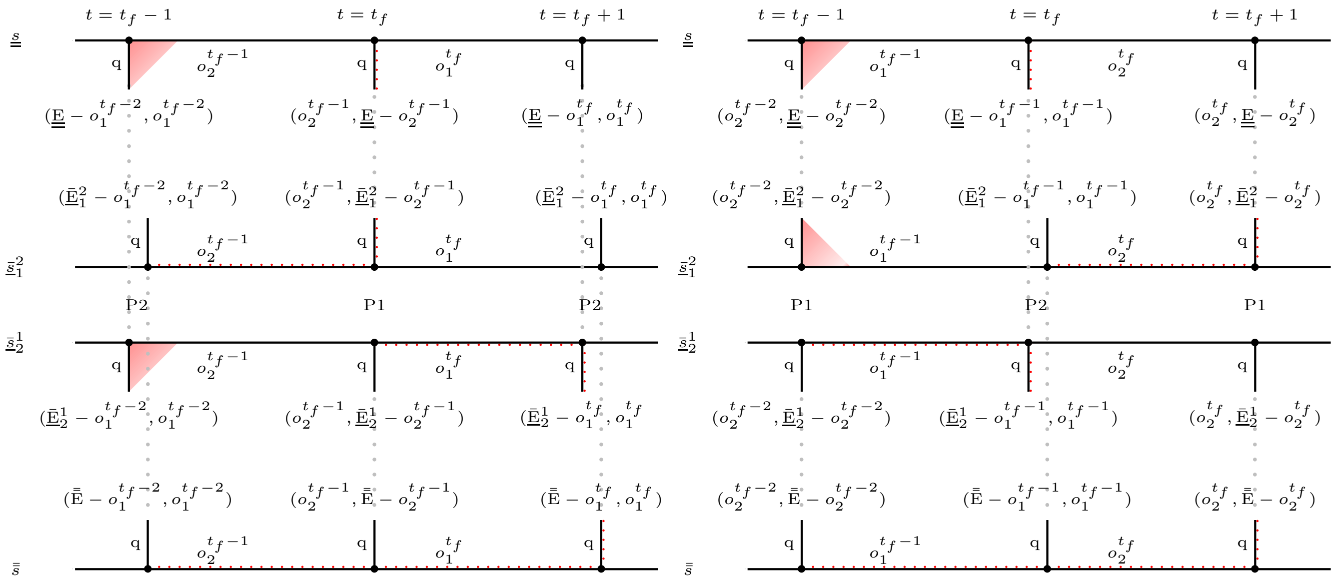

Figure 1: The endgame for odd t_f (left) and even t_f (right).

when Pi's minimal admissible bid $o_i^{t_f}$ is such that

$$\bar{\bar{E}} > o_i^{t_f} + o_j^{t_f-1} > \bar{\bar{E}} - 1. \tag{10}$$

Proof. That the action at t_f is separating follows from assumption (A). It fully reveals Pi's signal. Full revelation cannot be part of equilibrium prior to period t_f because, in equilibrium, after observing a continuation offer, Pj concludes that Pi's signal is high and thus, from assumption (A), continues wp1. But then Pi will continue wp1 *whatever* her type destroying the separating equilibrium candidate.

More precisely, consider a separating candidate $\hat{\beta}$ with $\underline{\mathrm{P}i}$ accepting wp1 at $\hat{t}$ although $o_i^{\hat{t}} + o_j^{\hat{t}-1} < \bar{\bar{\mathrm{E}}} - 1$ while $\overline{\mathrm{P}i}$ continues wp1. From subscribing to $\hat{\beta}$, $\underline{\mathrm{P}i}$ gets $o_j^{\hat{t}-1}$ with certainty. After observing a continuation from Pi, Pj must conclude that Pi's signal is high and continue wp1 because assumption (A) prescribes high-type continuation $o_j^{\hat{t}+1}$ wp1 even at the final period t_f. But then $\underline{\mathrm{P}i}$ can get $o_j^{\hat{t}+1}$ wp1 by continuing wp1 and accepting at $\hat{t}$ cannot be part of an equilibrium. □

We now show that high types will not accept wpp before period t_f and low types will not accept wp1 before t_f. Thus play reaches the terminal period t_f wpp.

Lemma 4. *In no equilibrium satisfying assumption (A), Pi accepts wp1 at period $\hat{t}$ if*

$$o_j^{\hat{t}-1} + o_i^{\hat{t}-2} < \bar{\bar{E}} - 2.$$

Proof. For the penultimate period $t_f - 1$, equilibrium mixing of $\underline{\mathrm{P}i}$ follows from assumption (A). Hence the high type moving at $t_f - 1$ continues wp1 because his beliefs are necessarily higher. Thus $\overline{\mathrm{P}j}$ will not want to accept wpp before period $t_f - 1$ because they can get at least $(o_i^{t_f})$ which is bigger than any previous offer in the game. Given this, $\overline{\mathrm{P}i}$ will not want to accept wpp earlier either. If $\underline{\mathrm{P}i}$ accepts wp1 at period $t_f - 2$ (or before) in equilibrium, her low signal is revealed which contradicts the previous lemma 3. □

The following lemma shows that both players' beliefs are uniquely determined on and off the equilibrium path. The reason is that, whenever Pi moves, it is profitable for $\overline{\mathrm{P}j}$ to induce Pi to accept immediately and it is beneficial to $\underline{\mathrm{P}j}$ to induce Pi to continue wp1. Hence every belief about the value of the object which does not make Pi exactly indifferent between accepting and continuing separates states. This is true at each stage where Pi can be made indifferent, ie. as soon as her priors do not induce an expected continuation payoff strictly above the current accepting payoff. Notice that Pi's main-game equilibrium mixture makes $\underline{\mathrm{P}j}$—whose mixing determines her belief—indifferent between accepting and continuing.

Lemma 5. *In every equilibrium satisfying assumption (A) and as long as* $o_j^{\hat{t}-1} + o_i^{\hat{t}-2} < \bar{\bar{E}} - 1$ *at* $\hat{t}$, *if* $\underline{\mathrm{P}i}$*'s prior belief* φ_i^1 *does not imply her continuation wp1, she must be indifferent between accepting and continuation. The low type's beliefs* $\underline{\varphi}_i^{\hat{t}}$ *uniquely determine those of the high-type player* $\bar{\varphi}_i^{\hat{t}}$.

Proof. a) Every equilibrium belief which would induce $\underline{\mathrm{P}j}$ to accept wp1 at $\hat{t}$ implies that $\underline{\mathrm{P}i}$ accepts wp1 at $\hat{t} - 1$ which is a fully separating action because we now from the previous lemma that $\overline{\mathrm{P1}}$ will not accept wpp. But we know from lemma 3 that there is no separating equilibrium when $o_i^{\hat{t}-1} + o_j^{\hat{t}-2} < \bar{\bar{\mathrm{E}}} - 1$.

b) Every belief which would induce Pj to continue wp1 at $\hat{t}$ implies that, in equilibrium, $\underline{\mathrm{P}i}$ continues wp1 at $\hat{t} - 1$ as well. Hence there is no updating of Pj's beliefs because $\overline{\mathrm{P}i}$ continues wp1 as well (lemma 4). But then $\underline{\mathrm{P}j}$'s beliefs are such that she prefers her continuation payoff to her acceptance payoff based on her unchanged prior belief $\varphi_j^{\hat{t}-2}$.

If $\underline{\mathrm{P}j}$ was indifferent between accepting and continuing based on the same prior at $\hat{t} - 2$, then she must prefer her necessarily increased acceptance payoff at $\hat{t}$, contradicting her continuing wp1. (Since both players continue wp1, $\underline{\mathrm{P}j}$'s continuation payoff is the same at $\hat{t}$ and $\hat{t} - 2$.) Hence she must have continued wp1 at $\hat{t} - 4$ as well and $\underline{\mathrm{P}i}$ did not mix at $\hat{t} - 5$ either. This argument is repeated until we reach Pj's prior φ_j^0.

c) Since high types continue wp1, the only information available for updating are the low types' mixtures. Hence the event which updates $\underline{\varphi}_i^t$ also updates $\bar{\varphi}_i^t$. □

The above lemma establishes an essentially unique belief-structure as part of β^*. This result generalises the corresponding result for incomplete information on one side and is a feature not commonly found in (repeated) signalling games. Thus the above allows us to calculate all equilibrium continuation payoffs prescribed by β^* provided that we can exclude jump-bids. This exclusion is the purpose of the following lemma 6.

Lemma 6. *In any equilibrium* β, *both players use minimal-increment strategies.*

Proof. Our argument proceeds by induction over time t, starting with the terminal equilibrium condition (9) without specifying offers. The idea is to start at *any* terminal node of the game where a player accepts wp1 in equilibrium and proceed backward—*keeping this terminal action fixed*—by choosing the stage action which maximises the payoff in the continuation game. Then, again *keeping the continuation game fixed*, we go a further period ahead and determine the optimal response to this fixed continuation game. Proceeding forward, this gives a unique equilibrium path through the game whatever terminal node we start at. We define a jump bid $j_i^t = o_i^t - o_i^{t-2} - 1$ as an offer which increases the last own offer by more than the minimal amount.

We start with the case where **a**) $\overline{\text{P1}}$ is making the final continuation bid (this corresponds to odd $\lfloor \bar{\bar{\text{E}}} \rfloor$ in equilibrium) and therefore reveals his signal. Assumption (A) implies that

$t_f + 1$: $\overline{\text{P2}}$ accepts with payoffs

$$u^{t_f+1}(q) = \left(\bar{\bar{\text{E}}} - o_1^{t_f}, o_1^{t_f}\right).$$

Hence we know that $o_2^{t_f-1} + o_1^{t_f} > \bar{\bar{\text{E}}}$. Similarly, $\underline{\text{P2}}$ accepts with payoffs

$$u^{t_f+1}(q) = \left(\underline{\bar{\text{E}}}_2^1 - o_1^{t_f}, o_1^{t_f}\right).$$

P2's accepting follows from the fact that we choose to analyse the equilibrium where $\overline{\text{P1}}$ continues up to his highest possible equilibrium bid. Then P2 must necessarily accept the following period.

t_f: $\overline{\text{P1}}$ continues with the above payoffs $u^{t_f+1}(q)$ and $\underline{\text{P1}}$ accepts with payoff $u_1^{t_f}(q) = o_2^{t_f-1}$.

$t_f - 1$: $\underline{\text{P2}}$ mixes and thus

$$o_1^{t_f-2} = (1 - \underline{\varphi}_2^{t_f-1})(\underline{\underline{\text{E}}} - o_2^{t_f-1}) + \underline{\varphi}_2^{t_f-1} o_1^{t_f}$$

implying that a minimal increase $j_2^{t_f-1} = 0$ is optimal. Mixing implies a terminal belief

$$\underline{\varphi}_2^{t_f-1} = \frac{o_1^{t_f-2} + o_2^{t_f-1} - \underline{\underline{\text{E}}}}{o_1^{t_f-2} + o_2^{t_f-1} - \underline{\underline{\text{E}}} + (o_1^{t_f} - o_1^{t_f-2})}. \tag{11}$$

Since $\underline{\text{P1}}$ mixes, $\overline{\text{P1}}$ continues wp1 getting

$$u_1^{t_f-1}(\underline{s}_2) = (1 - \bar{\varphi}_1^{t_f-1})(\underline{\bar{\text{E}}}_2^1 - o_1^{t_f-1}) + \bar{\varphi}_1^{t_f-1} o_2^{t_f}$$

which, again, implies $j_1^{t_f-1} = 0$.

$t_f - 2$: $\overline{\text{P1}}$ gets

$$u_1^{t_f-2}(\bar{s}_1) = (1 - \bar{\varphi}_1^{t_f-2})(\underline{\bar{\text{E}}}_2^1 - o_1^{t_f}) + \bar{\varphi}_1^{t_f-2}(\bar{\bar{\text{E}}} - o_1^{t_f}) = \bar{\text{E}}^1 - o_1^{t_f}$$

which is decreasing in $j_1^{t_f-2}$ and $\underline{\text{P1}}$ gets

$$u_1^{t_f-2}(\underline{s}_1) = (1 - \underline{\varphi}_1^{t_f-2})\left[(1 - \alpha_2^{t_f-1})(\underline{\underline{\text{E}}} - o_1^{t_f-2}) + \alpha_2^{t_f-1} o_2^{t_f-1}\right] + \underline{\varphi}_1^{t_f-2}(\underline{\bar{\text{E}}}_1^2 - o_2^{t_f-1})$$

which, similarly, is decreasing in $o_1^{t_f-2}$.

t_f-3: $\overline{\text{P2}}$ continues wp1 getting

$$u_2^{t_f-3}(\bar{s}_2) = (1-\bar{\varphi}_2^{t_f-3})\left[(1-\alpha_1^{t_f-2})(\bar{\underline{\text{E}}}_1^2 - o_2^{t_f-3}) + \alpha_1^{t_f-2} o_1^{t_f-2}\right] + \bar{\varphi}_2^{t_f-3} o_1^{t_f}$$

implying $j_2^{t_f-3} = 0$ for α_1^t defined in (1). Similarly, $\underline{\text{P2}}$ mixes and thus

$$u_2^{t_f-3}(\underline{s}_2) = (1-\bar{\varphi}_2^{t_f-3})\left[(1-\alpha_1^{t_f-2})(\underline{\underline{\text{E}}} - o_2^{t_f-3}) + \alpha_1^{t_f-2} o_1^{t_f-2}\right] + \bar{\varphi}_2^{t_f-3} o_1^{t_f-2}$$

which is decreasing in $o_2^{t_f-3}$.

t_f-4: $\overline{\text{P1}}$ gets

$$u_1^{t_f-4}(\bar{s}_1) = (1-\bar{\varphi}_1^{t_f-4})\left[(1-\alpha_2^{t_f-3})(\bar{\underline{\text{E}}}_2^1 - o_1^{t_f-4}) + \alpha_2^{t_f-3} o_2^{t_f-3}\right] + \bar{\varphi}_1^{t_f-4} o_1^{t_f}$$

which is decreasing in $j_1^{t_f-4}$ for α_2^t defined in (1). $\underline{\text{P1}}$ gets

$$u_1^{t_f-4}(\underline{s}_1) = (1-\bar{\varphi}_1^{t_f-4})\left[(1-\alpha_2^{t_f-3})(\underline{\underline{\text{E}}} - o_1^{t_f-4}) + \alpha_2^{t_f-3} o_2^{t_f-3}\right] + \bar{\varphi}_1^{t_f-4} o_2^{t_f-3}$$

which, similarly, is decreasing in $o_1^{t_f-4}$.

t: The argument is repeated unchanged for as long as both lower types mix.

b) Now consider the case where P2 is the player separating states at t_f. (This corresponds to even $\lfloor \bar{\bar{\text{E}}} \rfloor$ in equilibrium.) Assumption (A) implies

t_f+1: $\overline{\text{P1}}$ accepts with payoffs

$$u^{t_f+1}(q) = \left(o_2^{t_f}, \bar{\bar{\text{E}}} - o_2^{t_f}\right).$$

Hence we know that $o_1^{t_f-1} + o_2^{t_f} > \bar{\bar{\text{E}}}$. Similarly, $\underline{\text{P1}}$ accepts with payoffs

$$u^{t_f+1}(q) = \left(o_2^{t_f}, \bar{\underline{\text{E}}}_1^2 - o_2^{t_f}\right).$$

Again, P1's accepts because $\overline{\text{P2}}$ continues up to his highest possible equilibrium bid.

t_f: $\overline{\text{P2}}$ continues with the above payoffs $u^{t_f+1}(q)$ and $\underline{\text{P2}}$ accepts with payoff $u_2^{t_f}(q) = o_1^{t_f-1}$.

t_f-1: $\underline{\text{P1}}$ mixes and thus

$$o_2^{t_f-2} = (1-\underline{\varphi}_1^{t_f-1})(\underline{\underline{\text{E}}} - o_1^{t_f-1}) + \underline{\varphi}_1^{t_f-1} o_2^{t_f}$$

implying that a minimal increase $j_1^{t_f-1} = 0$ is optimal. Mixing implies a terminal belief

$$\underline{\varphi}_1^{t_f-1} = \frac{o_2^{t_f-2} + o_1^{t_f-1} - \underline{\underline{\text{E}}}}{o_2^{t_f-2} + o_1^{t_f-1} - \underline{\underline{\text{E}}} + (o_2^{t_f} - o_2^{t_f-2})}. \tag{12}$$

Since $\underline{\mathrm{P2}}$ mixes, $\overline{\mathrm{P2}}$ continues wp1 getting

$$u_2^{t_f-1}(\underline{s}_2) = (1-\bar{\varphi}_2^{t_f-1})(\underline{\bar{\mathrm{E}}}_1^2 - o_2^{t_f-1}) + \bar{\varphi}_2^{t_f-1} o_1^{t_f}$$

which, again, implies $j_2^{t_f-1} = 0$.

$t_f - 2$: $\overline{\mathrm{P2}}$ gets

$$u_2^{t_f-2}(\bar{s}_2) = (1-\bar{\varphi}_2^{t_f-2})(\underline{\bar{\mathrm{E}}}_1^2 - o_2^{t_f}) + \bar{\varphi}_2^{t_f-2}(\bar{\bar{\mathrm{E}}} - o_2^{t_f}) = \bar{\mathrm{E}}^2 - o_2^{t_f}$$

which is decreasing in $j_2^{t_f-2}$ and $\underline{\mathrm{P2}}$ gets

$$u_2^{t_f-2}(\underline{s}_2) = (1-\underline{\varphi}_2^{t_f-2})\left[(1-\alpha_1^{t_f-1})(\underline{\underline{\mathrm{E}}} - o_2^{t_f-2}) + \alpha_2^{t_f-1} o_1^{t_f-1}\right] + \underline{\varphi}_2^{t_f-2}(\underline{\bar{\mathrm{E}}}_2^1 - o_1^{t_f-1})$$

which, similarly, is decreasing in $o_2^{t_f-2}$.

$t_f - 3$: $\overline{\mathrm{P1}}$ continues wp1 getting

$$u_1^{t_f-3}(\bar{s}_1) = (1-\bar{\varphi}_1^{t_f-3})\left[(1-\alpha_2^{t_f-2})(\underline{\bar{\mathrm{E}}}_2^1 - o_1^{t_f-3}) + \alpha_2^{t_f-2} o_2^{t_f-2}\right] \bar{\varphi}_1^{t_f-3} o_2^{t_f}$$

implying $j_1^{t_f-3} = 0$ for α_2^t defined in (1). Similarly, $\underline{\mathrm{P1}}$ mixes and thus

$$u_1^{t_f-3}(\underline{s}_1) = (1-\bar{\varphi}_1^{t_f-3})\left[(1-\alpha_2^{t_f-2})(\underline{\underline{\mathrm{E}}} - o_1^{t_f-3}) + \alpha_2^{t_f-2} o_2^{t_f-2}\right] \bar{\varphi}_1^{t_f-3} o_2^{t_f-2}$$

which is decreasing in $o_1^{t_f-3}$.

$t_f - 4$: $\overline{\mathrm{P2}}$ gets

$$u_2^{t_f-4}(\bar{s}_2) = (1-\bar{\varphi}_2^{t_f-4})\left[(1-\alpha_1^{t_f-3})(\underline{\bar{\mathrm{E}}}_1^2 - o_2^{t_f-4}) + \alpha_1^{t_f-3} o_1^{t_f-3}\right] + \bar{\varphi}_2^{t_f-4} o_2^{t_f}$$

which is decreasing in $j_2^{t_f-4}$ for α_1^t defined in (1). $\underline{\mathrm{P2}}$ gets

$$u_2^{t_f-4}(\underline{s}_2) = (1-\bar{\varphi}_2^{t_f-4})\left[(1-\alpha_1^{t_f-3})(\underline{\underline{\mathrm{E}}} - o_2^{t_f-4}) + \alpha_1^{t_f-3} o_1^{t_f-3}\right] + \bar{\varphi}_2^{t_f-4} o_1^{t_f-3}$$

which, similarly, is decreasing in $o_2^{t_f-4}$.

t: Again, the argument can be repeated until a low type stops to mix when his acceptance payoff becomes less than his expected continuation payoff.

Any jump bid before the first mixture period of a low type reduces *all* own acceptance payoffs uniformly by the jump and reduces the opponent's final offer through shortening the game. Hence these cannot be profitable. The formal argument is identical to the case of incomplete information on one side.

Thus we have identified a path through the game starting at the period of the highest possible equilibrium of the game, through to the first period. Both players use minimal increase strategies on this path. As the path coincides with the prescription β^*, we have identified an equilibrium. Moreover, for the parameter region where the equilibrium exists, β^* is the essentially unique such equilibrium. $\square$

The next step determines all involved mixture probabilities uniquely.

Lemma 7. *Equilibrium mixture probabilities and beliefs are essentially uniquely determined.*

Proof. Given a high enough $\bar{\theta}$ and Pi's offer at odd t, $\underline{\text{P}j}$'s next period acceptance payoff is lower than her expected equilibrium continuation payoff for *any* belief $\varphi_j^{t+1} \in [{}^1\!/_2, 1]$ in the initial phase of play

$$u_j^{t+1}(q) < \underline{\varphi}_j^{t+1} u_j^{t+1}(\beta^*|\underline{\bar{s}}_j^i) + (1-\underline{\varphi}_j^{t+1}) u_j^t(\beta^*|\bar{\bar{s}}).$$

This means that $\underline{\text{P}j}$ will continue wp1 at $t+1$ and thus $\underline{\text{P}i}$ will not mix or else his expected payoffs are necessarily reduced. Since $\overline{\text{P1}}$ never mixes, Pj's posterior φ_j^{t+1} equals her conditional prior φ_j^1 in this initial phase.

Now consider any jump deviation from β^* by Pi which leaves Pj with a higher observed offer than expected. If there are beliefs $\varphi_j^{t+1} \in (\varphi_2^{t-1}, 1]$ for which it is possible that

$$u_j^{t+1}(q) > \underline{\varphi}_j^{t+1} u_j^{t+1}(\beta^*|\underline{\bar{s}}_j^i) + (1-\underline{\varphi}_j^{t+1}) u_j^t(\beta^*|\bar{\bar{s}}). \tag{13}$$

then $\underline{\text{P}i}$ must mix (ie. increase $\underline{\varphi}_j^{t+1}$) in order to make P$j$ indifferent between accepting and increasing her offer. Because there cannot be a separating equilibrium unless (10) holds (which means that the game is over after the next move), this is only possible if Pj is indifferent between her equilibrium continuation bid and accepting. Hence her belief $\underline{\varphi}_2^{t+1}$ is uniquely defined (as part of her equilibrium strategy) as the belief which makes (13) hold with equality. There is only one mixture probability which $\underline{\text{P}i}$ can use to generate these beliefs through his observed actions and Bayes' rule—and Pj has no choice but to assume that $\underline{\text{P}i}$ uses exactly this continuation probability. As soon as $\underline{\text{P}i}$ starts to mix, the equilibrium probabilities are obtained by inserting the equilibrium acceptance payoffs into the stage indifference conditions. For a jump bid $j_i^t = o_i^t - o_i^{t-2} - 1$ and running sum jump bids $J_i^t = \sum_{\hat{t}=i}^{(t+i)/2} j_i^{2\hat{t}-i}$, a players' time-$t$ offer—and thus their opponents' following period acceptance payoffs—are given through summation as

$$o_1^t = u_2^{t+1}(q) = \frac{t+1}{2} + J_1^t,\ o_2^t = u_1^{t+1}(q) = \frac{t}{2} + J_2^t. \tag{14}$$

Thus α_2^t is given, at even t, from $\underline{\text{P1}}$ mixing with any $\alpha_1^{t-1} \in (0,1)$ if $u_1^t(\beta^*|\underline{s}_1) = u_1^{t-1}(q)$ or

$$\begin{aligned} o_2^{t-2} &= (1-\underline{\varphi}_1^{t-1})\left[(1-\alpha_2^t)(\underline{\underline{\text{E}}} - o_1^{t-1}) + \alpha_2^t o_2^t\right] + \underline{\varphi}_1^{t-1} o_2^t \\ {}^*\alpha_2^t &= \frac{(1-\underline{\varphi}_1^{t-1})(o_1^{t-1} + o_2^t - \underline{\underline{\text{E}}}) - (o_2^t - o_2^{t-2})}{(1-\underline{\varphi}_1^{t-1})(o_1^{t-1} + o_2^t - \underline{\underline{\text{E}}})} \end{aligned} \tag{15}$$

which equals the proposed equilibrium prescription (1). Likewise, α_1^t is given for odd t from $\underline{\text{P2}}$

mixing with any $\alpha_2^{t-1} \in (0,1)$ iff $u_2^t(\beta^*|\underline{s}_2) = u_2^{t-1}(q)$, or

$$\begin{aligned} o_1^{t-2} &= (1-\underline{\varphi}_2^{t-1})\left[(1-\alpha_1^t)(\underline{\underline{\mathrm{E}}} - o_2^{t-1}) + \alpha_1^t o_1^t\right] + \underline{\varphi}_2^{t-1} o_1^t \\ {}^*\alpha_1^t &= \frac{(1-\underline{\varphi}_2^{t-1})(o_2^{t-1} + o_1^t - \underline{\underline{\mathrm{E}}}) - (o_1^t - o_1^{t-2})}{(1-\underline{\varphi}_2^{t-1})(o_2^{t-1} + o_1^t - \underline{\underline{\mathrm{E}}})} \end{aligned} \tag{16}$$

which again equals (1). Beliefs $\underline{\varphi}_i^t$ evolve according to Bayes' rule (2) implying terminal equilibrium beliefs (11) and (12). Hence on and off-equilibrium-path beliefs including Pj's final move are uniquely defined. As soon as $o_i^{t-1} + o_j^{t-2} \geq \bar{\bar{\mathrm{E}}} - 1$ at t, Pj accepts wp1 for any belief. Since then the previous *equilibrium* action by $\underline{\mathrm{P}i}$ is to accept wp1, Pj's belief is $\varphi_j = 1$ and thus uniquely defined as well. The only problem is after an off-equilibrium jump by Pi which leads Pj to accept for *any* belief. This is the reason for the qualification of β^* as 'essentially' unique. □

Proposition 1. *β^* defined in (3) is the essentially unique equilibrium of $\mathcal{Q}_B$ for a pair of suitably chosen signal precisions (p_1, p_2) satisfying assumption (A).*

Proof. Equilibrium existence is a direct consequence of the previous lemmata. Uniqueness follows from the fact that, in equilibrium, $\mathcal{Q}_B$ can *only end wp1* when (6) is reached by a high-type player. Although there are many possible histories leading to this condition, lemma 6 shows that only the minimal-increase profile is compatible with both (6) and maximisation at each stage. Lemma 7 supplies the essentially unique mixture probabilities and beliefs which turn the minimal-increase profile from lemma 6 into the equilibrium β^* for some period t_s^* to be determined in proposition 2. For the parameters we analyse, the only source of non-uniqueness of equilibria is at the terminal stage which we ignore and qualify the resulting equilibria as only 'essentially' unique. □

Combined, the above establishes existence and essential uniqueness of the equilibrium β^*. It does not involve jump bids. This involves some period t_s, when low types start to play mixed actions. The final step is to determine the period t_s where the first low type player starts to mix between accepting and continuing and to calculate the corresponding equilibrium payoffs.

Proposition 2. *In the equilibrium β^*, the first mixing period t_s^* is determined through the final mixing belief $\underline{\varphi}^{t_f-1}$. This pair of t_s^*, t_f^* pins down all expected payoffs of β^*.*

Proof. For odd $\lfloor \bar{\bar{\mathrm{E}}} \rfloor$, the evolution of $\underline{\mathrm{P2}}$'s equilibrium belief $\underline{\varphi}_2^t$ determines the first mixing period of the game. To see why this is the case, notice that—as argued in (11) of lemma 6—$\underline{\mathrm{P2}}$'s terminal equilibrium (path) mixing belief must be

$$\underline{\varphi}_2^{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - 1} = \frac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - \underline{\underline{\mathrm{E}}} - 1}{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - \underline{\underline{\mathrm{E}}}}.$$

Inserting this into Bayes' rule and substituting $\underline{\mathrm{P1}}$'s mixture probability for the equilibrium

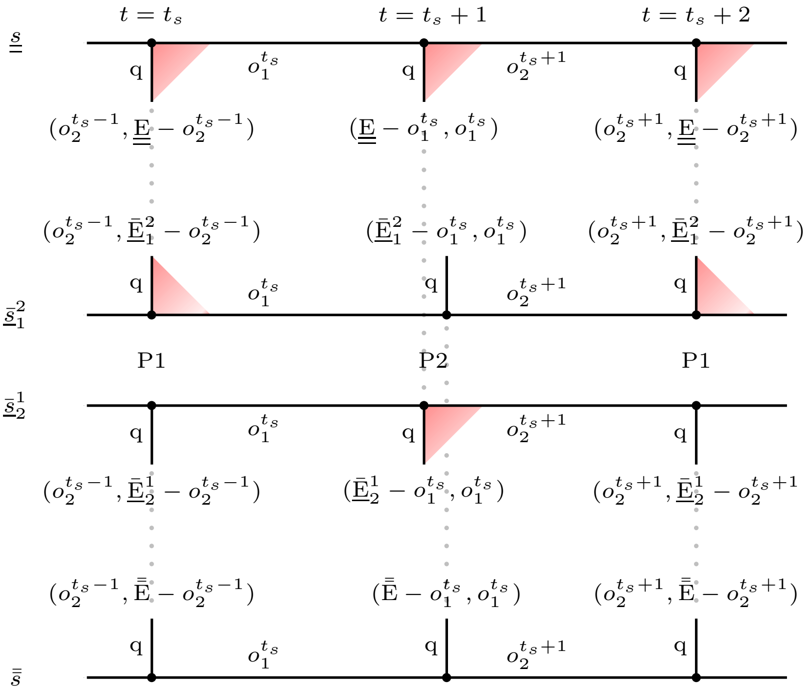

Figure 2: The end of the initial equilibrium phase at t_s for odd $\lfloor \bar{\bar{\mathrm{E}}} \rfloor$.

(1), one finds $\underline{\mathrm{P2}}$'s even time-t belief by calculating backwards as

$$ {}^*\underline{\varphi}_2^t = \left(\left(\left(\left(\frac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - \underline{\underline{\mathrm{E}}} - 1}{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - \underline{\underline{\mathrm{E}}}}\right) \frac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - \underline{\underline{\mathrm{E}}} - 3}{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - \underline{\underline{\mathrm{E}}} - 2}\right) \frac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - \underline{\underline{\mathrm{E}}} - 5}{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - \underline{\underline{\mathrm{E}}} - 4}\right) \dots\right)_{\frac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - t}{2} \text{ times}} = \prod_{\tau=1}^{\frac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - t + 1}{2}} \frac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - \underline{\underline{\mathrm{E}}} - 2\tau + 1}{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - \underline{\underline{\mathrm{E}}} - 2\tau + 2} $$

which, using Pochhammer notation $P(a, n) = \frac{\Gamma(a+n)}{\Gamma(a)}$, simplifies to

$$ {}^*\underline{\varphi}_2^t = \underline{\varphi}_2^{t+2} \frac{t - \underline{\underline{\mathrm{E}}}}{t - \underline{\underline{\mathrm{E}}} + 1} = \frac{P\left(\frac{1 - \lfloor \bar{\bar{\mathrm{E}}} \rfloor + \underline{\underline{\mathrm{E}}}}{2}, \frac{1 - \lfloor \bar{\bar{\mathrm{E}}} \rfloor - t}{2}\right)}{P\left(\frac{\underline{\underline{\mathrm{E}}} - \lfloor \bar{\bar{\mathrm{E}}} \rfloor}{2}, \frac{1 - \lfloor \bar{\bar{\mathrm{E}}} \rfloor - t}{2}\right)}. \tag{17} $$

We have found t_s^* as soon as this calculated ${}^*\underline{\varphi}_2^t$ becomes smaller than the prior $\underline{\varphi}_2^{t_s}$. Since a closed form solution to (17) for t_s is—to the author's knowledge—unavailable, we must use simulation techniques. This, however, does not impinge on our analytic existence argument and the characterisation of the equilibrium strategies. Now calculating from this t_s^* forward, $\underline{\mathrm{P1}}$'s final period belief is found from ${}^*\underline{\varphi}_1^t = \underline{\varphi}_1^{t-2} \frac{t - \underline{\underline{\mathrm{E}}}}{t - \underline{\underline{\mathrm{E}}} + 1}$ as

$$ \underline{\varphi}_1^{\lfloor \bar{\bar{\mathrm{E}}} \rfloor} = \underline{\varphi}_1^1 \frac{\Gamma\left(\frac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - \underline{\underline{\mathrm{E}}} - t_s + 4}{2}\right)}{\Gamma\left(\frac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - \underline{\underline{\mathrm{E}}} - t_s + 3}{2}\right)} \frac{\Gamma\left(\frac{t_s - \underline{\underline{\mathrm{E}}}}{2}\right)}{\Gamma\left(\frac{t_s - \underline{\underline{\mathrm{E}}} + 1}{2}\right)}. \tag{18} $$

The two resulting conditions (17) and (18) are the sufficient conditions for the existence of the equilibrium β^* as stated by assumption (A). The conditions for even $\lfloor \bar{\bar{\mathrm{E}}} \rfloor$ are identical for reversed roles of the players. Given the first mixture period t_s^*, the player's odd $\lfloor \bar{\bar{\mathrm{E}}} \rfloor$ expected payoffs are determined as

$$\begin{aligned}
u_1(\beta^*|\bar{\bar{s}}) =& \ \bar{\bar{\mathrm{E}}} - \tfrac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor + 1}{2}, \quad u_2(\beta^*|\bar{\bar{s}}) = \tfrac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor + 1}{2}, \\
u_1(\beta^*|\underline{\bar{s}}_2^1) =& \ \underline{\bar{\mathrm{E}}}_2^1 - u_2(\beta^*|\underline{\bar{s}}_2^1), \\
u_2(\beta^*|\underline{\bar{s}}_2^1) =& \sum_{\tau=\frac{t_s+1}{2}}^{\frac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - 1}{2}} \left(\prod_{t=\frac{t_s+1}{2}}^{\tau-1} \alpha_2^{2t} \right) u_2^{2\tau}(q)(1-\alpha_2^{2\tau}) + \left(\prod_{t=1}^{\frac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - 1}{2}} \alpha_2^{2t} \right) u_2^{\lfloor \bar{\bar{\mathrm{E}}} \rfloor}(q), \\
u_1(\beta^*|\bar{\underline{s}}_1^2) =& \sum_{\tau=\frac{t_s+1}{2}}^{\frac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor + 1}{2}} \left(\prod_{t=1}^{\tau-1} \alpha_1^{2t-1} \right) u_1^{2\tau-1}(q)(1-\alpha_1^{2\tau-1}) + \left(\prod_{t=\frac{t_s+1}{2}}^{\frac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor + 1}{2}} \alpha_1^{2t-1} \right) u_1^{\lfloor \bar{\bar{\mathrm{E}}} \rfloor}(q), \\
u_2(\beta^*|\bar{\underline{s}}_1^2) =& \ \bar{\underline{\mathrm{E}}}_1^2 - u_1(\beta^*|\bar{\underline{s}}_1^2), \quad u_1(\beta^*|\underline{\underline{s}}) = \underline{\underline{\mathrm{E}}} - u_2(\beta^*|\underline{\underline{s}}), \\
u_2(\beta^*|\underline{\underline{s}} =& \sum_{\tau=\frac{t_s+1}{2}}^{\frac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - 1}{2}} \left(\prod_{t=\frac{t_s+1}{2}}^{\tau-1} \alpha_2^{2t-1}\alpha_2^{2t} \right) \left((1-\alpha_1^{2\tau-1})u_2^{2\tau-1}(q) + \alpha_1^{2\tau-1}(1-\alpha_2^{2\tau})u_2^{2\tau}(q) \right).
\end{aligned}$$

Obviously, $u_1(\beta^*|\underline{s}_1) = \frac{t_s-1}{2}$. These payoffs can be directly computed by supplying the equilibrium continuation probabilities α_i^t and the corresponding beliefs φ_i^t from (2). For the case of even-valued $\lfloor \bar{\bar{\mathrm{E}}} \rfloor$, the expected payoffs are found in precisely the same way. □

The above proposition fully and essentially uniquely determines t_s^* but fails to find a closed form representation of t_s^*. Hence the resulting payoff characterisations are unwieldy and the payoff implications of varying the player's signal precisions are most easily determined in simulations. For sufficiently high $\bar{\theta}$ (ie. a not too coarse bidding grid), t_s^* is decreasing when the sum of available information $p_1 + p_2$ is going down and t_s^* is increasing when it is going up. This is intuitive as $\underline{\underline{\mathrm{E}}}$ and $\bar{\bar{\mathrm{E}}}$ move closer together with less information and further apart with more available information. As to be expected, Pi's payoff expectation from β^* moves in the same direction as p_i when holding p_j fixed.

5 Concluding remarks

We present the essentially unique solution to an alternating-offers bargaining problem where two players are asymmetrically informed about an object's common value. Extending the existing literature, we study the effects of the players mutually signalling their private information during the bargaining process. Extending the model to not necessarily increasing offers can be accomplished as in the model with incomplete information on one side and will not alter our results. Interesting generalisations of the model are involve a richer (perhaps continuous) signalling and/or type space, and a unified equilibrium parameterisation for any set of signal accuracies.

Appendix: Example

We first look at the simple example of $\theta \in \{0, 5\}$ for an arbitrarily chosen pair of signal accuracies $\mathbf{p} = (.8, .75)$. The set of possible bids is $\{0, 1, 2, 3, 4, 5, \ldots, \bar{B}\}$. The true value of the object is unknown to a player with signal precision $p < 1$ and thus there is some generic uncertainty on the object's value underlying the incomplete information on the opponent's signal. Therefore the true realisation of θ is generally not known to a player even if he were to know the opponent's signal: Learning the opponent's signal is the best a player can hope for. Thus we use signal profiles as states in figure 3 and not the realisation of θ.

The chosen signal accuracies give rise to the initial beliefs conditional on the own signal $\varphi_i^t = \mathrm{pr}(s_j = \bar{s}|s_i)$. These initial conditional beliefs are calculated from the common priors $\varphi_i^0 = {}^1\!/_2$ using Bayes' rule as, for instance, for $\underline{\mathrm{P2}}$

$$\begin{aligned}\underline{\varphi}_2^1 = \; & \mathrm{pr}(\underline{s}_1|\underline{s}_2) = \frac{\mathrm{pr}(\underline{s}_1, \underline{s}_2)}{\mathrm{pr}(\underline{s}_2)} = \frac{\mathrm{pr}(\underline{s}_1, \underline{s}_2|\underline{\theta})\,\mathrm{pr}(\underline{\theta}) + \mathrm{pr}(\underline{s}_1, \underline{s}_2|\bar{\theta})\,\mathrm{pr}(\bar{\theta})}{\mathrm{pr}(\underline{s}_2|\underline{\theta})\,\mathrm{pr}(\underline{\theta}) + \mathrm{pr}(\underline{s}_2|\bar{\theta})\,\mathrm{pr}(\bar{\theta})} = \\ & \frac{\mathrm{pr}(\underline{s}_1, \underline{s}_2|\underline{\theta}) + \mathrm{pr}(\underline{s}_1, \underline{s}_2|\bar{\theta})}{\mathrm{pr}(\underline{s}_2|\underline{\theta}) + \mathrm{pr}(\underline{s}_2|\bar{\theta})} = \frac{p_1p_2 + (1-p_1)(1-p_2)}{p_2 + (1-p_2)}.\end{aligned}$$

Filling in the example values of $\mathbf{p} = (.8, .75)$, the above give

$$\begin{aligned}\underline{\varphi}_1^1 &= p_1 + p_2 - 2p_1p_2 = .35, & \underline{\varphi}_2^1 &= (1-p_1)p_2 + p_1(1-p_2) = .35, \\ \bar{\varphi}_1^1 &= p_1p_2 + (1-p_1)(1-p_2) = .65, & \bar{\varphi}_2^1 &= (1-p_1)(1-p_2) + p_1p_2 = .65.\end{aligned}$$

Since the information on the signal accuracies is symmetric and public, the initial $\underline{\varphi}_1^1 = \underline{\varphi}_2^1$ and $\bar{\varphi}_1^1 = \bar{\varphi}_2^1$ must be identical given the same signal. Next we calculate the 'objective' expectation of the object's value given the different signal combinations. These are

$$\begin{aligned}\underline{\underline{\mathrm{E}}} &= \bar{\theta}\frac{(1-p_1)(1-p_2)}{(1-p_1)(1-p_2) + p_1p_2} = 0.38, & \bar{\underline{\mathrm{E}}}_1^2 &= \bar{\theta}\frac{(1-p_1)p_2}{p_1 + p_2 - 2p_1p_2} = 2.14, \\ \underline{\bar{\mathrm{E}}}_2^1 &= \bar{\theta}\frac{p_1(1-p_2)}{p_1 + p_2 - 2p_1p_2} = 2.86, & \bar{\bar{\mathrm{E}}} &= \bar{\theta}\frac{p_1p_2}{p_1p_2 + (1-p_1)(1-p_2)} = 4.62\end{aligned}$$

giving $\underline{\mathrm{P2}}$'s 'subjective' ex-ante expectation of the object's value as $\underline{\mathrm{E}}_2 = \underline{\varphi}_2^1\underline{\bar{\mathrm{E}}}_2^1 + (1 - \underline{\varphi}_2^1)\underline{\underline{\mathrm{E}}} = (1-p_2)\bar{\theta} = 1.25$.

Following the definition of the equilibrium β^* in section 3 we impose the low-type mixture conditions at each stage of the main game and obtain the following continuation probabilities from solving the resulting system of inequalities (which is derived below)

$$\beta_b^* = \begin{pmatrix} \underline{\alpha}_1^1 = 1 & \underline{\alpha}_2^2 = 0.21 & \underline{\alpha}_1^3 = 0.41 & \underline{\alpha}_2^4 = 0 & \underline{\alpha}_1^5 = 0 \\ \bar{\alpha}_1^1 = 1 & \bar{\alpha}_2^2 = 1 & \bar{\alpha}_1^3 = 1 & \bar{\alpha}_2^4 = 1 & \bar{\alpha}_1^5 = 0 \end{pmatrix} \tag{A.1}$$

together with the belief system

$$\begin{pmatrix} \underline{\varphi}_1^1 = .35 & \underline{\varphi}_2^2 = .35 & \underline{\varphi}_1^3 = .72 & \underline{\varphi}_2^4 = .57 \\ \bar{\varphi}_1^1 = .65 & \bar{\varphi}_2^2 = .65 & \bar{\varphi}_1^3 = .90 & \bar{\varphi}_2^4 = .82 \end{pmatrix}.$$

This candidate equilibrium path is marked red in the extensive form of figure 3 where shaded triangles symbolise mixed actions. To compute the above equilibrium profile, we weigh the low-type player's expected continuation payoffs from a profile β at $t \in \{1,2,3,4\}$

$$u_i^t(\beta|\underline{s}_i) = (1-\underline{\varphi}_i^t)\left[(1-\alpha_j^{t+1})u_i^{t+1}(q|\underline{s}_i,\underline{s}_j) + \alpha_j^{t+1}u_i^{t+2}(\beta|\underline{s}_i)\right] + \underline{\varphi}_i^t\left[u_i^{t+2}(\beta|\underline{s}_i)\right]$$

against the same player's *certain* acceptance payoff $u_i^t(q) = t + J_j$ as defined in lemma 2. (Where J_j is $j's$ running sum of jump bids.) These acceptance payoffs are independent of the object's expected value because they consist solely of the sum of the opponent's bidding increases. Notice that it is crucial for the easy solvability of the game that in the above continuation payoff $u_i^{t+2}(\beta|\underline{s}_i) = u_i^{t+2}(q|\underline{s}_i)$ because $\underline{\text{P}i}$ mixes at $t+2$. Of course, Pi is willing to mix only if the above accepting and continuation payoffs are equal.

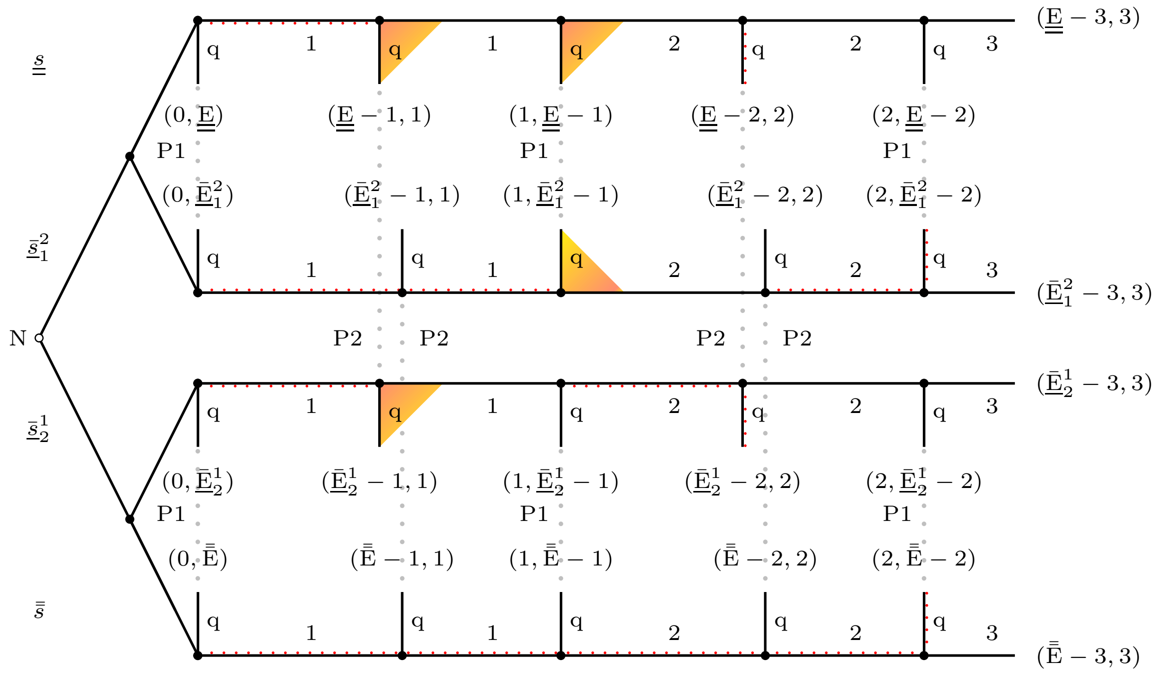

Figure 3: Partial extensive form for $\theta \in \{0,5\}$ and $\mathbf{p} = (.8,.75)$.

The equilibrium dynamics are as follows; first the *backward chain*: Since $\underline{\text{P2}}$ accepts wp1 rather than bidding 4, and $\overline{\text{P2}}$ bids 4 followed by $\overline{\text{P1}}$ accepting, the terminal belief $\underline{\varphi}_1^3$ which makes $\underline{\text{P1}}$ mix at $t=3$ is uniquely determined. There is only a single mixture probability α_2^2 which brings about these beliefs through Bayes' rule and thus this mixture probability is determined uniquely as well. The *forward chain* consists just of the indifference condition given the prior beliefs by $\underline{\text{P2}}$ at $t=2$ generating $\underline{\text{P1}}$'s mixture probability at $t=3$: $\underline{\text{P2}}$'s indifference between accepting and minimal continuation uniquely defines the α_1^3 which makes her mix. Notice that this does not include the mixing of $\underline{\text{P1}}$ at $t_f = 1$ (as in the case of odd $\bar{\bar{E}}$) because all probabilities are already uniquely determined. As prescribed by β^*, all high types continue wp1 until the minimal admissible continuation bids exceed $\lceil\bar{\bar{E}}\rceil$; then they quit.

Checking the player's equilibrium conditions amounts to setting up the system

$$\begin{array}{lll}
\text{low-type P1:} & t=1, & 0<(1-\underline{\varphi}_1^1)\left[(1-\underline{\alpha}_2^2)(\underline{\underline{\mathrm{E}}}-1)+\underline{\alpha}_2^2(1)\right]+\underline{\varphi}_1^1(1) \\
\text{high-type P1:} & t=1, & 0<(1-\bar{\varphi}_1^1)\left[(1-\underline{\alpha}_2^2)(\underline{\bar{\mathrm{E}}}_2^1-1)+\underline{\alpha}_2^2(1)\right]+\bar{\varphi}_1^1(2) \\
\text{low-type P2:} & t=2, & 1=(1-\underline{\varphi}_2^2)\left[(1-\underline{\alpha}_1^3)(\underline{\underline{\mathrm{E}}}-1)+\underline{\alpha}_1^3(2)\right]+\underline{\varphi}_2^2(2) \\
\text{high-type P2:} & t=2, & 1<(1-\bar{\varphi}_2^2)\left[(1-\underline{\alpha}_1^3)(\underline{\bar{\mathrm{E}}}_1^2-1)+\underline{\alpha}_1^3(2)\right]+\underline{\varphi}_2^2(\bar{\bar{\mathrm{E}}}-2) \\
\text{BR:} & t=3, & \underline{\varphi}_1^3=\mathrm{pr}(\bar{s}_2|\underline{s}_1,b_2^2=2)=\dfrac{\underline{\varphi}_1^1}{\underline{\varphi}_1^1+(1-\underline{\varphi}_1^1)\underline{\alpha}_2^2},\bar{\varphi}_1^3=\dfrac{\bar{\varphi}_1^1}{\bar{\varphi}_1^1+(1-\bar{\varphi}_1^1)\underline{\alpha}_1^1} \\
\text{low-type P1:} & t=3, & 1=(1-\underline{\varphi}_1^3)(\underline{\underline{\mathrm{E}}}-2)+\underline{\varphi}_1^3(2) \\
\text{high-type P1:} & t=3, & 1<(1-\bar{\varphi}_1^3)(\underline{\bar{\mathrm{E}}}_2^1-2)+\bar{\varphi}_1^3(2) \\
\text{BR:} & t=4, & \underline{\varphi}_2^4=\mathrm{pr}(\bar{s}_1|\underline{s}_2,b_1^3=3)=\dfrac{\underline{\varphi}_2^2}{\underline{\varphi}_2^2+(1-\underline{\varphi}_2^2)\underline{\alpha}_1^3},\bar{\varphi}_2^4=\dfrac{\bar{\varphi}_2^2}{\bar{\varphi}_2^2+(1-\bar{\varphi}_2^2)\underline{\alpha}_1^3} \\
\text{low-type P2:} & t=4, & 2>(1-\underline{\varphi}_2^4)(\underline{\underline{\mathrm{E}}}-2)+\bar{\varphi}_2^4(\underline{\bar{\mathrm{E}}}_2^1-2) \\
\text{high-type P2:} & t=4, & 2<(1-\bar{\varphi}_2^4)(\underline{\bar{\mathrm{E}}}_1^2-2)+\bar{\varphi}_2^4(\bar{\bar{\mathrm{E}}}-2) \\
\text{BR:} & t=5, & \underline{\varphi}_1^5=\mathrm{pr}(\bar{s}_2|\underline{s}_1,b_2^4=4)=\dfrac{\underline{\varphi}_1^3}{\underline{\varphi}_1^3+(1-\underline{\varphi}_1^3)\underline{\alpha}_2^4}=1,\bar{\varphi}_1^5=\dfrac{\bar{\varphi}_1^3}{\bar{\varphi}_1^3+(1-\bar{\varphi}_1^3)\underline{\alpha}_2^4}=1 \\
\text{low-type P1:} & t=5, & 2>(\underline{\bar{\mathrm{E}}}_1^2-3) \\
\text{high-type P1:} & t=5, & 2>(\bar{\bar{\mathrm{E}}}-3)
\end{array}$$

which is solved by (A.1). This, together with an unsuccessful search for deviations,[9] confirms β^* with probabilities (A.1) as equilibrium of our example. Its expected payoffs are

	$(\underline{s}_1,\underline{s}_2)$	$(\underline{s}_1,\bar{s}_2)$	$(\bar{s}_1,\underline{s}_2)$	$(\bar{s}_1,\bar{s}_2)$	$(\underline{s}_1,\cdot)$	$(\bar{s}_1,\cdot)$	$(\cdot,\underline{s}_2)$	$(\cdot,\bar{s}_2)$	$\mathbb{E}[\cdot]$
$u_1(\beta^*\vert s)$	−.50	1.41	1.65	2.00	0.17	1.88	·	·	1.02
$u_2(\beta^*\vert s)$	0.89	0.73	1.21	2.62	·	·	1.00	1.96	1.48

As pointed out previously, it is not possible to implement β^* for all pairs of signal accuracies **p**. As sufficiency condition for existence of β^*, assumption (A) demands that

$$0.8185=\bar{\varphi}_2^4>\frac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor-\underline{\bar{\mathrm{E}}}_1^2}{\bar{\bar{\mathrm{E}}}-\underline{\bar{\mathrm{E}}}_1^2}=0.7511$$

which ensures that the $\overline{\mathrm{P2}}$ continues at period $\lfloor\bar{\bar{\mathrm{E}}}\rfloor=4$. It is fulfilled for the chosen signal accuracies. Likewise, $\underline{\mathrm{P1}}$'s period t_f-1 belief allows for his mixing and the grid condition $\lfloor\bar{\bar{\mathrm{E}}}\rfloor>\underline{\bar{\mathrm{E}}}_2^1$ holds.

To illustrate a deviation, suppose P2 observes $\hat{b}_1^1=2$ instead of the equilibrium-prescribed

[9] In order to confirm that there are no profitable deviations we need to work out the players' on- and off-equilibrium path beliefs. This is done in accord with lemmata 3 and 5. What these lemmata say is that both low-type players must be indifferent between accepting and continuing after each feasible deviation (ie. after each deviation which does not force subsequent accepting wp1). At the same time, a player's equilibrium strategy states a mixture probability for every history. Since only a single belief is compatible with using that equilibrium response, beliefs are fully determined.

$b_1^1 = 1$; then her *equilibrium* mixture condition at $t = 2$ turns into

$$2 = (1 - \hat{\underline{\varphi}}_2^2)(\underline{\underline{\mathrm{E}}} - 1) + \hat{\underline{\varphi}}_2^2(3) \quad \Leftrightarrow \quad \hat{\underline{\varphi}}_2^2 = \frac{3 - \underline{\underline{\mathrm{E}}}}{4 - \underline{\underline{\mathrm{E}}}} = 0.72 = \frac{\underline{\varphi}_2^1(1)}{\underline{\varphi}_2^1(1) + (1 - \underline{\varphi}_2^1)\hat{\alpha}_1^1}$$

resulting in the requirement of P2 believing that the low-type P1's deviation occurred with probability $\hat{\alpha}_1^1 = 0.21$. *Any* other belief will result in P2 either continuing or accepting for certain meaning that P1 could manipulate P2 into doing what is optimal for him. Since P1 would do this *whatever his type*, this cannot be equilibrium behaviour.

The lesson from our example is threefold: (*i*) Players cannot 'lie' to their opponent by playing supposedly fully revealing actions because the opponent would not believe such dubious information. (*ii*) This renders jump-bidding unprofitable because it ties the deviating player to offering a higher-than-equilibrium share to the equilibrium player until agreement is reached. (*iii*) The only way of using private information is to gradually release it by playing partially revealing, type-dependent mixed actions until all information is transferred.

General p

In this subsection we present a fully worked example for the case of $\theta \in \{0, 5\}$. In principle, our problem is to find areas (ie. parameterised equilibria) in $p_1 \times p_2$ demarcated by our equilibrium conditions (which are polynomial inequalities). These conditions are simply the preference of the continuation payoff over the acceptance payoff or vice versa and indifference between the two for mixed actions. Imposing these conditions at each stage of the game gives a region of the corresponding information requirements in $p_1 \times p_2$. As $\bar{\theta}$ becomes large, however, these conditions become numerous and of increasingly high order and hence solving for the resulting systems of equilibrium conditions becomes difficult even for state of the art computer math packages.[10]

Since we are looking for *all* full-dimension equilibria now,[11] we have to consider both low- and high-signal mixed actions the continuation probability of which we denote by $\underline{\alpha}$ and $\bar{\alpha}$. To denote strategy profiles we use matrices where, for instance

$$\begin{pmatrix} 1 & 1 & 1 & m \\ 1 & m & m & 0 \end{pmatrix} \text{ represents the profile } \begin{pmatrix} \bar{\alpha}_1^1 = 1 & \bar{\alpha}_2^2 = 1 & \bar{\alpha}_1^3 = 1 & \bar{\alpha}_2^4 \in (0,1) \\ \underline{\alpha}_1^1 = 1 & \underline{\alpha}_2^2 = \in (0,1) & \underline{\alpha}_1^3 = \in (0,1) & \underline{\alpha}_2^4 = 0 \end{pmatrix}.$$

The time-5 continuation probability α_1^5 is zero for any $\mathbf{p}$ and any signal and is therefore omitted. In accord with our model specification, the signal accuracy $\mathbf{p} = (p_1, p_2)$ specifies the probability

[10] The field concerned with the general analytic study of such problems is that of *algebraic geometry*. However, even using specialised computer software developed for the study of numerical algebraic geometry problems we were unable to compute results for cases where $\bar{\theta}$ is significantly larger than in the present example. For a survey of the methods and techniques involved see Baxter and Iserles (2003).

[11] We call solutions 'full-dimension' equilibria if the associated precision pair is in the interior of the parameter set in $p_1 \times p_2$ for which the equilibrium is defined. The complementary 'measure zero' equilibria are knife-edge parameter cases at the boundaries of these sets. Since the boundaries of these sets describe mixed actions, we typically disregard final stage, high-type mixtures in the present analysis.

with which the received signal is correctly indicating the true value of the object. The possible range for the publicly known (asymmetric) idiosyncratic signal precision p_i is $[^1/_2, 1],\ i = \{1, 2\}$ where $p_i = {}^1/_2$ means that Pi gets no additional information on top of her priors and $p_i = 1$ means that her signal is fully revealing. Hence the case with incomplete information on one side is described by $\mathbf{p} = (1, 0)$. Matrices such as the one above represent systems of polynomial inequalities solved by a system of restrictions on the constants α and $\mathbf{p}$. These results are summarised in figure 4. In the following we list the strategies for which solutions in $\mathbf{p}$ (ie. parameterised equilibrium candidates) can be found. We sort them according to the number of pure low-type continuation actions the strategies contain. These parameterised solutions involve recurring conditions $f_1(\cdot)$–$f_3(\cdot)$ for mixed actions which are defined for convenience in the legend of figure 4.

The remarkable result is that there is a unique map of full-dimension parameterised equilibria which covers $p_1 \times p_2$. Hence the essential uniqueness result of the analysis of the case of incomplete information on one side is preserved in this particular example of incomplete information on both sides for *any* parameter combination.

Discussion

The map in figure 4 has two main features: Firstly ,the strategy profiles (A.2)–(A.14) fully cover the parameter space $p_1 \times p_2$ and, secondly, there is an *unique* equilibrium for any $\mathbf{p}$ in full dimension.[12] The equilibria are intuitively appealing. For instance in the lower-left-corner equilibrium region (A.13), the players have very little information and cannot effectively discriminate between the high and low signal states. Hence they bid up to the expectation of the object and accept as soon as the required bid exceeds this expectation in a (near-)pooling strategy. As expected from the analysis of the case of incomplete information on one side, (A.3), the essentially unique equilibrium of that case can be retrieved in the more general setting of incomplete information on both sides. For $\alpha_1^5 = 0$, it occupies the line segment $p_1 \in [^4/_5, 1]$ for $p_2 = {}^1/_2$. The equilibrium β^* discussed in the main section and in last subsection's example for $\mathbf{p} = (.8, .75)$ is confirmed by (A.9). The map shows both equilibria in fully revealing (separating) and non-revealing strategies: In (A.8), for instance, $\underline{\text{P2}}$ reveals her type at t=2 by accepting. In what follows, we identify and solve for all relevant strategy profiles of figure 4.

No pure low-type continuation

$$\begin{pmatrix} 1 & m & 1 & m \\ m & 0 & m & 0 \end{pmatrix} \Rightarrow {}^4/_5 \le p_1 \le 1 \wedge 0 \le p_2 \le \frac{3p_1 - 4}{p_1 - 3} \text{ for} \tag{A.2}$$

$$\underline{\alpha}_1^1 = \tfrac{7-25p_1+25p_1^2}{3-20p_1+25p_1^2},\ \bar{\alpha}_2^2 = \tfrac{5p_1-4}{5p_1-3},\ \underline{\alpha}_1^3 = \tfrac{5p_1-4}{5\underline{\alpha}_1^1 p_1 - \underline{\alpha}_1^1},\text{ and } \bar{\alpha}_2^4 = \tfrac{5p_1-2}{5p_1-1}.$$

[12] There are more parameterised equilibria of measure zero but we disregard them in the present discussion. There are no other equilibria in full dimension $\mathbf{p}$.

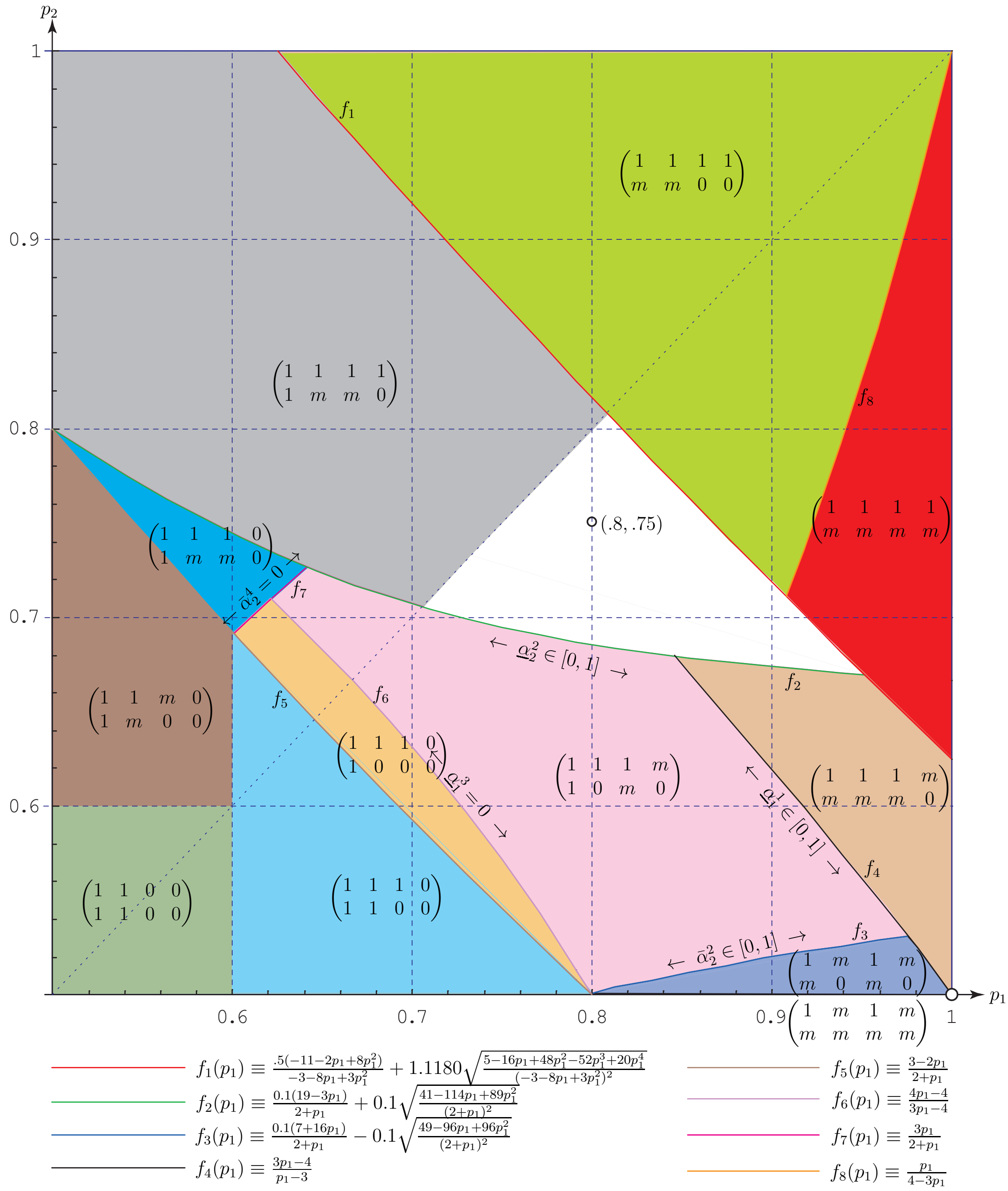

Figure 4: The equilibrium map for $\theta \in \{0, 5\}$ in $p_1 \times p_2$ signal accuracy space; the example area β^* is left white.

$$\begin{pmatrix} 1 & m & 1 & m \\ m & m & m & m \end{pmatrix} \Rightarrow {}^4/_5 \leq p_1 \leq 1 \wedge p_2 = {}^1/_2 \tag{A.3}$$

This is the essentially unique equilibrium of the case of incomplete information on one side. It is not directly comparable to the other equilibria because of the implicit constraint that $\alpha_2^t = \bar{\alpha}_2^t = \underline{\alpha}_2^t$ for all t stemming from the fact that P1 has perfect information in that model. It is a measure zero equilibrium with $\underline{\alpha}_1^1 = \frac{7-25p_1+25p_1^2}{3-20p_1+25p_1^2}$, $\alpha_2^2 = \frac{5p_1-4}{5p_1-3}$, $\underline{\alpha}_1^3 = \frac{5p_1-4}{5\underline{\alpha}_1^1 p_1 - \underline{\alpha}_1^1}$, and $\alpha_2^4 = \frac{5p_1-2}{5p_1-1}$.

$$\begin{pmatrix} 1 & 1 & 1 & m \\ m & m & m & 0 \end{pmatrix} \Rightarrow \begin{cases} (i) & 0.8453 \le p_1 \le 0.9515 \wedge \frac{3p_1-4}{p_1-3} \le p_2 \le f_2(p_1) \\ (ii) & 0.9515 < p_1 \le 1 \wedge \frac{3p_1-4}{p_1-3} \le p_2 \le f_1(p_1) \end{cases} \text{for} \tag{A.4}$$

$\underline{\alpha}_2^2 = \frac{4-3p_1-3p_2+p_1p_2}{3-3p_1-3p_2+p_1p_2}$, $\underline{\alpha}_1^3 = \frac{-3\underline{\alpha}_1^1-p_1+3\underline{\alpha}_1^1p_1-p_2+3\underline{\alpha}_1^1p_2+2p_1p_2-\underline{\alpha}_1^1p_1p_2}{-2\underline{\alpha}_1^1+2\underline{\alpha}_1^1p_1+2\underline{\alpha}_1^1p_2+\underline{\alpha}_1^1p_1p_2}$, and $\bar{\alpha}_2^4 = \frac{2\underline{\alpha}_2^2-3p_1-2\underline{\alpha}_2^2p_1+2p_2-2\underline{\alpha}_2^2p_2+p_1p_2-\underline{\alpha}_2^2p_1p_2}{-4p_1+p_2+3p_1p_2}$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ m & m & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} (i) & 5/8 \le p_1 \le .9078 \wedge f_1(p_1) \le p_2 \le 1 \\ (ii) & 0.9078 < p_1 < 1 \wedge \frac{p_1}{4-3p_1} \le p_2 \le 1 \end{cases} \text{for} \tag{A.5}$$

$\underline{\alpha}_2^2 = \frac{4-3p_1-3p_2+p_1p_2}{3-3p_1-3p_2+p_1p_2}$ and $\underline{\alpha}_1^1 = \frac{-p_1-p_2+2p_1p_2}{3-3p_1-3p_2+p_1p_2}$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ m & m & m & m \end{pmatrix} \Rightarrow f_1(p_1) \le p_2 \le \frac{p_1}{4-3p_1} \wedge 0.9078 \le p_1 \le 1 \tag{A.6}$$

One pure low-type continuation

$$\begin{pmatrix} 1 & 1 & m & 0 \\ 1 & m & 0 & 0 \end{pmatrix} \Rightarrow 1/2 \le p_1 \le 3/5 \wedge 3/5 \le p_2 \le \frac{3-2p_1}{2+p_1} \text{ for} \tag{A.7}$$

$\underline{\alpha}_2^2 = \frac{3-3p_1-3p_2+p_1p_2}{2p_1-3p_2+p_1p_2}$ and $\bar{\alpha}_1^3 = \frac{3-5p_2}{2p_1-3p_2+p_1p_2}$.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} (i) & 3/5 \le p_1 \le 0.6202 \wedge \frac{3-2p_1}{2+p_1} \le p_2 \le \frac{3p_1}{2+p_1} \\ (ii) & 0.6202 < p_1 \le 4/5 \wedge \frac{3-2p_1}{2+p_1} \le p_2 \le \frac{-4+4p_1}{-4+3p_1} \end{cases}. \tag{A.8}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & m & m & 0 \end{pmatrix} \Rightarrow \begin{cases} (i) & 1/2 \le p_1 \le 3/5 \wedge \frac{3-2p_1}{2+p_1} \le p_2 \le f_2(p_1) \\ (ii) & 3/5 < p_1 \le 16/25 \wedge \frac{3p_1}{2+p_1} \le p_2 \le f_2(p_1) \end{cases} \text{for} \tag{A.9}$$

$\underline{\alpha}_2^2 = \frac{-3p_1+2p_2+p_1p_2}{-2+2p_1+2p_2+p_1p_2}$ and $\underline{\alpha}_1^3 = \frac{-3+2p_1+2p_2+p_1p_2}{-2+2p_1+2p_2+p_1p_2}$.

$$\begin{pmatrix} 1 & 1 & 1 & m \\ 1 & 0 & m & 0 \end{pmatrix} \Rightarrow \begin{cases} (i) & 0.6202 \leq p_1 \leq 0.64 \wedge \frac{4p_1-4}{3p_1-4} \leq p_2 \leq {}^{3p_1}/_{2p_1} \\ (ii) & 0.64 < p_1 \leq {}^4/_5 \wedge \frac{4p_1-4}{3p_1-4} \leq p_2 \leq f_2(p_1) \\ (iii) & {}^4/_5 < p_1 \leq .8453 \wedge f_3(p_1) \leq p_2 \leq f_2(p_1) \\ (iv) & .8453 < p_1 < 0.9756 \wedge f_3(p_1) \leq p_2 \leq \frac{3p_1-4}{p_1-3} \end{cases} \quad \text{for} \tag{A.10}$$

$\underline{\alpha}_2^4 = \frac{-3p_1+2p_2+p_1p_2}{4p_1+2p_2+3p_1p_2}$ and $\underline{\alpha}_1^3 = \frac{4-4p_1-4p_2+3p_1p_2}{-4p_1+p_2+3p_1p_2}$.

$$\begin{pmatrix} 1 & 1 & 1 & m \\ 1 & m & m & 0 \end{pmatrix} \Rightarrow \begin{cases} (i) & p_1 = {}^1/_2 \wedge p_2 = {}^4/_5 \wedge \bar{\alpha}_2^4, \underline{\alpha}_2^2, \underline{\alpha}_1^3 \\ (ii) & p_1 = 0.55 \wedge p_2 = 0.77 \wedge \bar{\alpha}_2^4, \underline{\alpha}_2^2, \underline{\alpha}_1^3 \\ (iii) & p_1 = 0.60 \ \wedge p_2 = 0.7444 \wedge \bar{\alpha}_2^4, \underline{\alpha}_2^2, \underline{\alpha}_1^3 \\ (iv) & p_1 = 0.95 \wedge p_2 = 0.6698 \wedge \bar{\alpha}_2^4, \underline{\alpha}_2^2, \underline{\alpha}_1^3 \\ (v) & p_1 > 0.95 \Rightarrow \emptyset \end{cases} . \tag{A.11}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & m & m & 0 \end{pmatrix} \Rightarrow \begin{cases} (i) & {}^1/_2 \leq p_1 <= {}^5/_8 \wedge f_2(p_1) \leq p_2 \leq 1 \\ (ii) & {}^5/_8 < p_1 < {}^4/_5 \wedge f_2(p_1) \leq p_2 \leq f_1(p_1) \\ (iii) & {}^5/_8 < p_1 \leq 0.9515 \wedge f_2(p_1) \leq p_2 \leq f_1(p_1) \end{cases} \quad \text{for} \tag{A.12}$$

$\underline{\alpha}_2^2 = \frac{p_1+p_2-2p_1p_2}{-2+2p_1+2p_2+p_1p_2}$ and $\underline{\alpha}_1^3 = \frac{-3+2p_1+2p_2+p_1p_2}{-2+2p_1+2p_2+p_1p_2}$.

This is the equilibrium β^* discussed in the main text.

Two pure low-type continuations

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \Rightarrow {}^1/_2 \leq p_1 \leq {}^3/_5 \wedge {}^1/_2 \leq p_2 \leq {}^3/_5. \tag{A.13}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \Rightarrow {}^3/_5 \leq p_1 \leq {}^4/_5 \wedge {}^1/_2 \leq p_2 \leq \frac{3-2p_1}{2+p_1}. \tag{A.14}$$

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