All-pay-all aspects of political decision making*

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Abstract

Decision-making processes are studied using non-standard all-pay structures. Our interest is motivated by regulatory, political, legal, military, and economic applications in which individual actions determine the consequences for a larger group or the general public. The common features of these examples are a competitive environment, a winner-takes-all reward structure, and some form of all-pay-all payment rule. (JEL *C7, D7, L5.* Keywords: *Auctions, Contests, Regulation, Conflict.*)

1 Introduction

Many strategic situations can be understood as contests in which participants compete for payoffs. Contests have a unifying 'all-pay' property: participants' efforts or investments are sunk, regardless of winning or losing. The most widely studied payment structure in contests is the 'first-price' rule, whereby players pay their own cost of effort, regardless of winning. Applications include lobbying, sports contests, political competition, labor tournaments, and many other first-past-the-post competitions. Another widespread class of contest applications exhibits a 'second-price' structure, whereby all participants pay their own bid except for the winner (or winners), who pays the highest losing effort. These games are often described as 'wars of attrition' because, to win the prize, the winner need only match or 'overcome' the best rival's effort. Typically, the literature models both classes of applications as all-pay auctions or contests either with or without an additional element of 'luck.'

Recently there has been more interest in winner-takes-all contests with alternative payment rules. The literature has argued that variants and generalizations of the standard all-pay auction may find applications in many new areas, including legal systems or behavioral economics.[1] Our paper adds to this discussion, arguing that there are important applications that fit the framework of all-pay auctions but are not captured by either of the widely studied first- or second-price payment rules. Our contribution is to study two novel classes of n-player all-pay auction games.

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[1] See, for instance, the examples in Baye, Kovenock, and de Vries (1998, 2012).

The first class of all-pay auctions relates to regulatory problems. Consider a symmetric oligopoly in which firms' profits are regulated using a price cap. In order to reduce industry profits, the regulator may run a 'proposal game' in which the oligopolists are obliged to simultaneously submit proposals for reducing all firms' profits by implementing a corresponding reduction in the price cap. Specifically, a firm's bid is made in terms of profit and is interpreted as a proposed uniform reduction of existing profits. Thus, through the bid, the firm proposes uniformly to reduce each firm's profit. The regulator commits to implementing the most stringent proposal. Provided that not all firms submit zero bids, the game results in a reduction in the oligopolists' profits.[2] We assume that each firm has a positive and privately known valuation of winning the game, i.e., of making the winning proposal. This value of winning can, for instance, be denominated in terms of publicity payoffs or brand-name capital.[3] Hence, we focus on a private-values, all-pay auction in which only winning is valuable and all players pay the winner's bid. We show that such a game is effective in reducing the industry profit in non-cooperative equilibrium.

Another application is voting games, where the issue at hand is a reduction in, e.g., pay or bonuses of the participating players. Following unfavorable press coverage, for instance, board members may be asked to propose cuts in their annual bonuses. Suppose that shareholders or a board of supervisors commit to implementing the most stringent reduction proposed.[4] Again, these games implement a payoff reduction in equilibrium where the most stringent reduction in compensation is proposed by the player who values winning the most.

Our second class of all-pay auctions can be motivated as follows. Recall that the classic war of attrition is traditionally illustrated by two players fighting for access to a resource. The fight ends as soon as one player gives in, which implies that both players pay the loser's effort cost. It is natural to extend this model by taking into account that one player's effort imposes a negative externality on the other player, e.g., in the form of injury or destruction. Thus, a player's effort choice produces a direct effect in the form of effort cost and an indirect effect in the form of injury to the rival. This results in both players paying (a function of) total effort.[5]

Depending on the specific application, our auctions may be games within larger games such that the participation decision depends on more than the direct payoff from the auction. Hence, for both classes of games, we explicitly consider the role of outside options.[6] For instance, a firm in a regulated industry must abide by the rules or face fines. Similarly, for a board member, participating

[2] In this game, the players' bids are interpreted as proposals, rather than efforts. Only the winning proposal is ex post payoff relevant.

[3] The EU commission's recent regulation of mobile phone roaming charges may serve as an example, documented in, for example, King (2011). The payoff from being perceived as consumer friendly may be the motivation behind mobile carrier Vodafone's announcement of the 'travel promise' (the statement of 3-Aug-2005 begins with 'Commitment to simplify and increase value to Vodafone customers') and competitor T-Mobile's 'Un-carrier' strategy, both of which can be seen as widely advertised announcements of 'consumer-friendly' price reductions in order to generate future payoffs.

[4] Three examples from politics, business, and the public sector, respectively, are Kasperowicz (2013), Chazan (2013), and Foster (2010).

[5] This idea is not new. The two-player complete information case of this setup has recently been analyzed by Baye, Kovenock, and de Vries (2012). In particular, see their section 'Territorial contests with injuries.' Moreover, Baye, Kovenock, and de Vries (1998) specify a 'civil war' parametrization of their general contest class, which is a special case of our auction model.

[6] Higgins, Shughart, and Tollison (1985) provide the first analysis of endogenous participation in contests.

in a vote is obligatory; abstention is inconsequential in the sense that the vote's outcome is binding on all members regardless of participation.

Literature

Our paper contributes to the study of all-pay auction games.[7] Some recent and related contributions include Krishna and Morgan (1997), Hopkins and Kornienko (2007), Kaplan, Luski, Sela, and Wettstein (2002), and Klose and Kovenock (2011a). Krishna and Morgan (1997) study a generalized war of attrition, as well as the first-price, all-pay auction, both with affiliated valuations. They provide revenue-ranking results for these and other sealed-bid formats. Both Hopkins and Kornienko (2007) and Kaplan, Luski, Sela, and Wettstein (2002) explore non-standard pricing rules that differ from ours. Klose and Kovenock (2011a) is a study under complete information. Klose and Kovenock (2011b) investigate a setup related to ours—that of identity-dependent externalities—but their analysis and research questions are set within a voting model. Güth and van Damme (1986) and Amann and Leininger (1996) discuss hybrids between asymmetric versions of the first- and second-price sealed-bid, all-pay auctions.[8] In these hybrid formats, the loser pays its bid while the winner pays a convex combination of its own and the loser's bid. For recent progress on contests with externalities, see Maasland and Onderstal (2007) or, again, Klose and Kovenock (2011a).

The papers closest to our analysis are Goeree and Turner (2000) and Baye, Kovenock, and de Vries (1998, 2005, 2012). The first paper introduces the all-pay-all auction, in which all bidders pay a weighted sum of all bids. The authors show that this is the optimal format when all bidders receive some share of the auction revenue.[9] In these auctions, bidders have an incentive to drive up the price. In contrast, players in our games do not benefit from other players' bids.

Baye, Kovenock, and de Vries (1998) is a precursor to two papers by Baye, Kovenock, and de Vries (2005, 2012) in which the authors study a general parameterized class of two-player contests. A special case of their model coincides with a special case of ours. In particular, their parameters $(\alpha, \beta, \delta, \theta, \gamma)$ have the values $(0, \alpha, 0, \alpha, 0)$ in our setting.[10] Their results coincide with ours as follows. If we evaluate their result (2) for the independent private-values case and a loser's prize of $\gamma = 0$, then (2) is equivalent to our equilibrium bid functions (10) and (12) for the two-player case and linear payments ($k = 1$, in the second game).

[7] The beautiful theory of these games has been developed, among others, by Dasgupta (1986), who discusses R&D races, Hillman and Samet (1987), who introduce rent-seeking and lobbying contests, Hillman and Riley (1989) and Baye, Kovenock, and de Vries (1993), who analyze competitions for monopoly, Clark and Riis (1998), who study the competition for promotions, and Moldovanu and Sela (2001), who discuss optimal prize structures. The availability of recent and comprehensive surveys of the contests literature such as Garfinkel and Skaperdas (2006), Congleton, Hillman, and Konrad (2007), Corchón (2007), and Konrad (2008) allows us to focus our discussion on only the most directly related parts of the literature.

[8] For further discussions of asymmetric all-pay auctions, see, for instance, Hillman and Riley (1989), Fibich, Gavious, and Sela (2006), Parreiras and Rubinchik (2010), or Szech (2011). For a detailed discussion, we appeal, again, to the above-mentioned literature surveys.

[9] A later published variant, Goeree, Maasland, Onderstal, and Turner (2005), is less related to the present work because, there, the payment rules do not exhibit an 'all-pay-all' structure.

[10] The parameters $(\alpha, \beta, \delta, \theta, \gamma)$ appear in the winner's and loser's linear payoff functions in Baye, Kovenock, and de Vries (1998, 2012). In Baye, Kovenock, and de Vries (2005), the parameters are essentially the same, with $\gamma = 0$.

Baye, Kovenock, and de Vries (2005) study dispute-resolution systems in a two-player game of incomplete information. The authors analyze so-called fee-shifting rules that are relevant in the legal context. We study n-player games, but, for the case of two players, our paper is complementary as follows. Our first game corresponds to $(\alpha, \beta, \delta, \theta) = (0, \alpha, 0, \alpha)$ in Baye et al.'s notation. This parameter constellation violates their assumptions $\alpha > 0$ and (A3) 'internalized legal cost,' which rules out subsidizing or taxing legal expenditures. For (our) $\alpha = 1$, the parametrization is equivalent to a legal system in which the winner bears its own cost while the loser pays a tax that raises the loser's cost to the level of the winner's cost. This idea is similar in nature to the Quayle system analyzed by Baye, Kovenock, and de Vries (2005), which attempts to discourage abuse of the court system by compelling the loser to make payments exceeding the loser's own cost.[11] Our second game corresponds to $(\alpha, \beta, \delta, \theta) = (\alpha, \alpha, \alpha, \alpha)$ if we set $k = 1$ (linear payment) in our model. This parameter cluster is consistent with the assumptions in Baye, Kovenock, and de Vries (2005) for $\alpha = 1/2$. Thus, our paper contains an analysis of the 'legal system' ($\alpha = 1/2, \beta = 1/2$), where every party pays one-half of the total legal expenditures. Moreover, variation of our parameters α and k allows us to study (hypothetical) legal systems wherein litigation costs are subsidized or taxed, compelling the parties to make (potentially nonlinear) payments to the court that are higher or lower than their actual legal expenditures. Note that, similarly, the results of Baye, Kovenock, and de Vries (1998) apply directly to hypothetical legal systems with linear payments where players' expenditures are taxed or subsidized.

The present paper fits into the general framework of contests with 'rank-order dependent spillovers,' as introduced by Baye, Kovenock, and de Vries (2012). These are contests for which players' efforts, in addition to being costly, imply external effects on other players' payoffs. This class of games covers a wide range of well-known problems. Our results are complementary to theirs, but while Baye, Kovenock, and de Vries (2012) study two-player games of complete information, we analyze two restricted classes of n-player games of incomplete information. In particular, our first game corresponds to their setup for $(\alpha, \beta, \delta, \theta, \gamma) = (0, \alpha, 0, \alpha, 0)$, whereas our second game corresponds to $(\alpha, \beta, \delta, \theta, \gamma) = (\alpha, \alpha, \alpha, \alpha, 0)$ if we set our parameters to $k = 1$ (linear payoff) in the second game, and $n = 2$ players in both games. In their terminology, our games for $n = 2$ exhibit both a 'first-order negative direct effect' and a 'second-order negative spillover effect.' In addition, our second game has a 'first-order negative spillover effect' and a 'second-order negative direct effect.'

Bulow and Klemperer (1999) analyze an incomplete information, dynamic, generalized war of attrition in which being active is costly; players may continue to pay a cost even after quitting the game. They interpret the latter 'strategic independence' case as a 'standards battle.' This is because, until a standard emerges, all players suffer losses regardless of whether or not they are actively promoting their own preferred alternative.

Lizzeri and Persico (2000) study uniqueness and existence of equilibria in a general class of auction

[11] So called after former US Vice President Dan Quayle, who chaired a commission that recommended a reform of the US legal system. According to this proposal, the loser should reimburse the winner's legal costs up to the amount actually spent by the loser. The Quayle system is one of the legal systems analyzed in Baye, Kovenock, and de Vries (1998, 2005).

problems. This class includes mechanisms closely related to those we study (see, for example, their example 5, a combination of first-price, all-pay auction and the war of attrition). However, their assumption (A7) excludes our setup. It says, inter alia, that the loser's payoff must not depend on the winner's bid.

The remainder of this paper is structured as follows: Following the definition of the environment in the next section, we provide examples to illustrate our main results. The general analysis is presented in section 3, followed by a concluding discussion of our findings. Proofs can be found in the appendix.

2 The model and examples

There are $n \geq 2$ risk-neutral and ex-ante identical bidders. Each bidder i has a privately known valuation θ_i for winning the game. Valuations θ_i are independent and identically distributed according to the continuously differentiable cumulative distribution function F, with support (normalized to) $[0, 1]$. We assume that F admits a continuous density $f(\theta_i) = F'(\theta_i)$ with full support.

We study simultaneous auction games of incomplete information. Each bidder i chooses a bid $b_i \in [0, \infty)$. The solution concept is Bayesian Nash equilibrium. A bidding strategy β is a mapping from types θ_i into bids b_i, $\beta : [0, 1] \to \mathbb{R}_+$. A zero bid is interpreted as non-participation. In this case, a bidder's payoff is equal to the symmetric, commonly known and perhaps negative outside option $\bar{u} \in (-\infty, +\infty)$. For $b_i > 0$, bidder i's payoff is

$$\pi(b_1, \ldots, b_n|\theta_i) = \begin{cases} \theta_i - h(b_1, \ldots, b_n) & \text{if } b_i > \max_{j \neq i}\{b_j\} \\ -h(b_1, \ldots, b_n) & \text{if } b_i < \max_{j \neq i}\{b_j\} \\ \frac{\theta_i}{m} - h(b_1, \ldots, b_n) & \text{if } b_i = \max_{j \neq i}\{b_j\}, \end{cases} \tag{1}$$

where m is the number of bidders concurrently issuing the highest bid.[12] Hence, the bidder who submits the highest positive bid wins the auction and realizes his or her private value θ_i. All other bidders receive nothing. All *participating* bidders make the same payment $h(b_1, \ldots, b_n)$, regardless of whether they win or lose. In the first class of auctions we study, $h(b_1, \ldots, b_n) = \alpha \max\{b_1, \ldots, b_n\}$, $\alpha > 0$, while in the second class of games the payment is $h(b_1, \ldots, b_n) = \alpha \sum_{j=1,\ldots,n} b_j^k$, where $\alpha, k > 0$.

Order statistics are employed, which are denoted as follows. The highest of $n - 1$ independent draws from the cumulative distribution function F is $\Theta_{(1:n-1)}$. Its distribution is denoted by G, with corresponding probability density g.

2.1 Examples of the two auction formats

Full participation of all bidder types requires sufficiently small outside options. Otherwise, there is generally a critical type below which players will not participate in the auction. For the purpose of presenting a simple example, we assume here that outside options are sufficiently small to ensure

[12] Ties have zero probability in the equilibria we derive.

full participation in equilibrium. (In the general analysis of section 3, we characterize the role of outside options in detail.)

Recall the regulatory 'price cap' game from the introduction. The regulator asks two symmetric duopolists simultaneously to submit proposals for a new and lower price cap. The regulator commits to implementing the lowest proposed price cap. We do not explicitly model the price cap.[13] Rather, a firm's bid in this auction is equal to the profit reduction implied by the introduction of the proposed price cap. As argued in the introduction, firms may value being known as the proposer of consumer-friendly regulation. Thus, a firm's payoff in this regulatory game is equal to a positive valuation element in the event of winning, θ_i, minus the profit reduction (bid) proposed by the winning bidder. For instance, with bids $b_1 < b_2$, the regulator would implement a profit reduction of amount b_2, with resulting payoffs $u_1 = -b_2$ and $u_2 = \theta_2 - b_2$.

Taking the existence of a symmetric, monotone bidding function β as given, player i of type $\theta_i \in [0,1]$ chooses her bid b_i solving

$$\max_{b_i} \int_0^{\beta^{-1}(b_i)} (\theta_i - b_i) f(\theta_j) d\theta_j - \int_{\beta^{-1}(b_i)}^1 \beta(\theta_j) f(\theta_j) d\theta_j. \tag{2}$$

Taking the derivative with respect to b_i, we obtain the first-order condition

$$f(\beta^{-1}(b_i))(\theta_i + \beta(\beta^{-1}(b_i)) - b_i)\frac{d\beta^{-1}(b_i)}{db_i} - \int_0^{\beta^{-1}(b_i)} f(\theta_j)\, d\theta_j = 0 \tag{3}$$

which, after invoking symmetry, $\beta^{-1}(b_i) = \theta_i$ and $\frac{d\beta^{-1}(b_i)}{db_i} = \frac{1}{\beta'(\theta_i)}$, reduces to

$$\beta'(\theta_i) = \frac{f(\theta_i)}{F(\theta_i)}\theta_i. \tag{4}$$

By integrating both sides of (4) over $(0, \theta_i]$ and using the boundary condition $\beta(0) = 0$, we obtain the equilibrium bidding function[14]

$$\int_0^{\theta_i} \beta'(x) dx = \int_0^{\theta_i} \frac{f(x)x}{F(x)} dx \iff \beta(\theta_i) = \int_0^{\theta_i} \frac{f(x)x}{F(x)} dx. \tag{5}$$

For this example, we restrict attention to the class of distribution functions $F(s) = s^t$, for any $t > 0$. This includes concave and convex functions. Then $\beta(\theta_i) = t\theta_i$, which, for the special case of uniform valuations, $F(\theta) = \theta$, implies truth-telling, $\beta(\theta_i) = \theta_i$. As there is a value to winning, proposing 'zero reduction' of the price cap is not an equilibrium strategy. In the litigation framework of Baye, Kovenock, and de Vries (2005), this example corresponds to the following legal system: the winner pays its own legal costs, whereas the loser pays a tax to the court such that legal expenditures are raised to the winner's cost level.[15] In section 3.1, we extend this example to n players, such that every one of them pays the same *multiple* of the highest bid.

[13] In a simple symmetric oligopoly model, the price cap is proportional to the firm's profits. In an asymmetric setting, the firms' bids can be understood as proposals for direct lump-sum reductions of profits.

[14] Given full participation, this boundary condition is intuitive: if the valuation is zero, then winning has no upside. The zero type would focus on minimizing payments: by making the lowest feasible bid, the price reduction in the event of winning is minimized.

[15] The case of taxation is mentioned in Baye, Kovenock, and de Vries (2005) as a way of reducing wasteful legal expenditures.

Consider now our second class of problems. There is a battle between two players wherein the 'bid' is fighting effort. Exerting this effort generates a cost for the player. At the same time, it has a direct negative effect on the rival, perhaps in the form of injury. A positive effort on the part of both players implies that each pays a function of both bids. The player who expends the higher effort wins the battle. As a simple example, consider the two-player, all-pay auction in which each player pays the *average* bid. Then, stipulating full participation and the existence of a symmetric, monotone equilibrium bidding function β, player i solves

$$\begin{aligned} &\max_{b_i} \quad \int_0^{\beta^{-1}(b_i)} \left(\theta_i - \frac{b_i + \beta(\theta_j)}{2}\right) f(\theta_j)d\theta_j - \int_{\beta^{-1}(b_i)}^1 \frac{b_i+\beta(\theta_j)}{2} f(\theta_j)d\theta_j \quad \Longleftrightarrow \\ &\max_{b_i} \quad F(\beta^{-1}(b_i))\theta_i - \tfrac{1}{2}\left(b_i + \int_0^1 \beta(\theta_j)f(\theta_j)d\theta_j\right). \end{aligned} \tag{6}$$

Taking the derivative with respect to b_i, we get

$$f(\beta^{-1}(b_i))\frac{d\beta^{-1}(b_i)}{db_i}\theta_i - \frac{1}{2} = 0. \tag{7}$$

After invoking symmetry, this reduces to $\beta(\theta_i)' = 2f(\theta_i)\theta_i$. Clearly, a bidder with zero valuation optimally bids zero. Therefore, with boundary condition $\beta(0) = 0$, the bid function is

$$\beta(\theta_i) = 2\int_0^{\theta_i} f(x)x\,dx. \tag{8}$$

We have chosen the 'average bid' payment rule, because (8) is equal to (4) in Baye, Kovenock, and de Vries (2005) for their parameters $\alpha = \beta = \delta = \theta = 1/2$. Our example corresponds to a legal system whereby each party pays the average legal expenditure, resulting in twice the *actual* equilibrium legal expenditure of the American system (in which all litigants pay their own legal costs). Baye, Kovenock, and de Vries (2005) show that a tradeoff exists between the American system and the British system (under which the loser pays its own costs and, in addition, reimburses the winner for all legal costs): the American system exhibits higher *expected* payoffs for litigants (implying incentives in favor of litigation), whereas the British system results in higher total *expected* legal expenditures per trial (reducing litigation incentives). On both measures, our example regime lies between the American and the British ones, representing a compromise.[16] In section 3.2, we analyze a more general n-player version of the present examples, where bidders pay a multiple of a (non)linear function of all bids.

[16] Baye, Kovenock, and de Vries (2005) show that these performance measures are a function of their β only, with $\beta = 1$ and $\beta = 0$ in the American and British systems, respectively, while our example uses $\beta = 1/2$. Apart from this case, we cannot compare results because, as soon as we deviate from $\alpha = 1/2$ and linear payoffs, we violate assumption (A3) in Baye, Kovenock, and de Vries (2005), which is incorporated in their solution from the beginning. Moreover, note that these results of Baye, Kovenock, and de Vries (2005) are comparable with ours given only that there is litigation, i.e., participation.

3 Analysis

3.1 Everyone pays (a multiple of) the highest bid

In the following, we analyze the all-pay auction game in which all participating bidders pay the same constant multiple $\alpha > 0$ of the highest bid. The highest bid wins the auction. We study the class of symmetric, pure-strategy equilibria with continuous, strictly monotone bid functions conditional on participation. In the unique equilibrium of this class, bidders participate (and make positive bids) if their valuations exceed a cutoff value, denoted by $\hat{\theta}$. This cutoff depends on the common outside option $\bar{u}$ and implies full (no) participation for sufficiently small (large) outside options. Unfortunately, this cutoff value cannot be stated in closed form, so we define it implicitly, using the equilibrium interim expected payoff of a bidder with valuation $\theta_i \geq \hat{\theta}$, denoted by $\Pi(\hat{\theta}, \theta_i)$. Our first result pins down a boundary condition for the equilibrium bidding function.

Lemma 1. *In any symmetric, monotone, pure-strategy equilibrium of the specified all-pay auction,* $\beta(\hat{\theta}) = 0$.

Proposition 1. *Consider the all-pay auction with symmetric outside option $\bar{u} \in (-\infty, \infty)$, in which all bidders pay the multiple $\alpha > 0$ of the highest bid. This auction game has an equilibrium that is unique in the class of symmetric, continuous, strictly monotone (conditional on participation), and pure-strategy equilibria. Define, for $\theta_i \geq \hat{\theta}$,*

$$\Pi(\hat{\theta}, \theta_i) = 1 - (n-1) \int_{\hat{\theta}}^{1} \frac{f(x)x}{F(x)}\, dx - \int_{\theta_i}^{1} G(x)\, dx. \tag{9}$$

In equilibrium, there is full participation if $\bar{u} \leq \Pi(0,0)$ and no participation if $\bar{u} \geq \Pi(1,1) = 1$. Otherwise, the cutoff valuation $\hat{\theta}$, above which a player participates, is implicitly defined by $\bar{u} = \Pi(\hat{\theta}, \hat{\theta})$. The equilibrium bidding function is, for participating bidders $\theta_i \geq \hat{\theta}$, defined as

$$\beta(\theta_i) = \frac{n-1}{\alpha} \int_{\hat{\theta}}^{\theta_i} \frac{f(x)x}{F(x)}\, dx, \tag{10}$$

where $\hat{\theta} = 0$ if $\bar{u} \leq \Pi(0,0)$.

Whenever at least one player participates in the auction, then the player who has the highest valuation wins. However, if all valuations are below the cutoff, then the player with the highest valuation does not win even though this player might value winning more than the outside option. Therefore, the auction is not ex post efficient. Observe that a player's interim expected equilibrium payoff (9) is independent of α. In this sense, the exact payment rule is immaterial within the class of rules we study. Thus, by varying α we can manipulate the size of equilibrium bids. This includes over- as well as underbidding one's valuation in equilibrium, and, interestingly, also 'truthful bidding' for all types, which we can show for a specific class of distribution functions: suppose that $F(\theta) = \theta^t$ for some $t > 0$. Then, in the symmetric equilibrium, all players bid their valuations, $\beta(\theta_i) = \theta_i$, for sufficiently small outside options, $\bar{u} \leq -\alpha^2/(1+\alpha)$, and α and n such that $\alpha/(n-1) = t$. In order to see this, set $F(\theta) = \theta^{\alpha/(n-1)}$ to get $\theta f(\theta)/F(\theta) = \alpha/(n-1)$. Then (10) directly simplifies to $\beta(\theta_i) = \theta_i$.[17] As mentioned in our literature review, for the two-player case, our result (10) is implied by (2) of Baye, Kovenock, and de Vries (1998).[18] Note that—being exogenous—α can be a function of n, leaving room for other interpretations and applications. For instance, consider a proposal game in which only the winning proposal is implemented and is costly. Then $\alpha = 1/n$ implies that all bidders pay an equal share of the cost of the implemented proposal.

3.2 The price as a function of all bids

We now turn to our second set of applications, modelled as an all-pay-all auction wherein all participating bidders pay the multiple $\alpha > 0$ of (a function of) all bids. The highest bid wins the auction. Players obviously do not participate if their outside option exceeds $\bar{u} = 1$. For sufficiently low outside options, there is full participation. Otherwise, in every symmetric, monotone, pure-strategy bidding equilibrium, there exists a cutoff value, denoted by $\hat{\theta}$, above which all player types participate in the auction. As in our first auction, this cutoff value cannot be stated in closed form, so we define it implicitly, using the equilibrium interim expected payoff of a bidder with valuation $\theta_i \geq \hat{\theta}$, denoted by $\Pi(\hat{\theta}, \theta_i)$.

Proposition 2. *Consider the all-pay auction with symmetric outside option $\bar{u} \in (-\infty, \infty)$ in which every bidder pays a function of all bids, $\alpha \sum_{j=1}^{n} (b_j)^k$, with $\alpha, k > 0$. This auction game has an equilibrium that is unique in the class of symmetric, continuous, strictly monotone (conditional on participation), and pure-strategy equilibria. Define, for $\theta_i \geq \hat{\theta}$,*

$$\Pi(\hat{\theta}, \theta_i) = G(\hat{\theta})\hat{\theta} + \int_{\hat{\theta}}^{\theta_i} G(x)\, dx - (n-1) \int_{\hat{\theta}}^{1} (1 - F(x)) g(x) x\, dx. \tag{11}$$

In equilibrium, there is full participation if $\bar{u} \leq \Pi(0,0)$ and no participation if $\bar{u} \geq \Pi(1,1) = 1$. Otherwise, the cutoff valuation $\hat{\theta}$, above which a player participates, is implicitly defined by $\bar{u} = \Pi(\hat{\theta}, \hat{\theta})$. The equilibrium bidding function is, for participating bidders $\theta_i \geq \hat{\theta}$, defined as

$$\beta(\theta_i) = \left(\frac{1}{\alpha} \int_{\hat{\theta}}^{\theta_i} g(x) x\, dx \right)^{\frac{1}{k}}, \tag{12}$$

where $\hat{\theta} = 0$ if $\bar{u} \leq \Pi(0,0)$.[19]

As in the previous section, the auction is not ex post efficient. The interim expected payoffs, (11), are independent of the parameters α and k. Hence, the specific payment rule (within the class we study) affects the size of equilibrium bids, but leaves expected payoffs unchanged. Again, by changing α and k we can implement over- and underbidding of the players' valuations. We can also

[17] Similarly, replace $f(x)x/F(x)$ in (9) by $\alpha/(n-1)$, simplify, and solve $\bar{u} = \Pi(0,0)$ to obtain the required upper bound of the outside option.

[18] We cannot compare results with Baye, Kovenock, and de Vries (2005), however, because we violate their assumption (A3) 'Internalized legal cost,' on which their relevant results rest.

[19] Recall that $g(x) = G'(x) = (n-1)F^{n-2}(x)f(x)$.

show examples of truthful bidding in equilibrium.[20]

Recall our summary of the relevant literature. Baye, Kovenock, and de Vries (1998) use a related parametrization that they refer to as 'civil war.' In their notation, we analyze $(\alpha, \beta, \delta, \theta, \gamma) = (\alpha, \alpha, \alpha, \alpha, 0)$ for $n \geq 2$ players and $\alpha > 0$, while the 'civil war' model is $(1, 1, 1, 1, 0)$ for $n = 2$ (and our $k = 1$, linear payments). Moreover, for $n = 2$, our result can be used to study legal systems with a structure similar to the classes studied in Baye, Kovenock, and de Vries (1998, 2005), including systems in which legal expenditure is taxed or subsidized. Unlike these existing results, our analysis features n players and the parameter k allows for nonlinear payments.

3.3 Discussion

The two classes of all-pay auction games that are discussed in the two preceding sections are closely related, as can be seen already in (1). Our results share much of the economic intuition with standard (all-pay) auctions. In particular, both games do not have equilibria wherein all players submit 'very low' bids. Moreover, our results extend to asymmetric outside options, provided they are commonly known to be sufficiently small to ensure full participation.

This section develops some intuition for our results. This is done by highlighting similarities with the standard first-price all-pay auction (called 'FP-APA' below). The intuition is clearer in the second game. Therefore, consider the second auction and, for simplicity, ignore the issue of participation. Denoting the opponents' random valuations by Θ, bidder i's expected interim payoff can be written as

$$\pi(\theta_i) = G(\theta_i)\theta_i - \alpha\beta^k(\theta_i) - (n-1)\,\mathbb{E}[\alpha\beta(\Theta)^k]. \tag{14}$$

For $\alpha = k = 1$, the first two terms are equal to the FP-APA payoff, while the remaining third term is constant with respect to i's valuation, and therefore does not affect incentives. Thus, intuitively, bidding behavior should be similar to that in the FP-APA auction.[21] Indeed, by comparison with (12), we have $\alpha\beta(\theta_i)^k = \beta^{\text{FP-APA}}(\theta_i)$. While bidding is similar to that in the FP-APA (corrected for α and k), payoffs are lower because one must also pay everyone else's bids (the last term in (14)).[22]

Now, consider the first auction (again ignoring the issue of participation). Bidder i's expected interim payoff can be written as

$$\pi(\theta_i) \;= G(\theta_i)\theta_i - \alpha\beta(\theta_i) - (1 - G(\theta_i))\left[\mathbb{E}[\alpha\beta(\Theta_{(1:n-1)})|\Theta_{(1:n-1)} > \theta_i] - \alpha\beta(\theta_i)\right). \tag{15}$$

[20] Consider full participation and $F(\theta) = \theta^t$, for some $t > 0$, and for which the auction parameters are $k = t(n-1)+1$ and $\alpha = (k-1)/k$, then the equilibrium exhibits truth-telling $\beta(\theta_i) = \theta_i$: $F(x) = x^t$ implies that $g(x)x = t(n-1)x^{t(n-1)}$. Inserting this into (12) and setting the term equal to θ_i (truthful bidding), we get

$$\theta_i \;= \left(\frac{1}{\alpha}\int_0^{\theta_i}\left(x^{t(n-1)}\right)'x\,dx\right)^{\frac{1}{k}} \;\Rightarrow\; \alpha\theta_i^k = \frac{t(n-1)}{t(n-1)+1}\theta_i^{t(n-1)+1}. \tag{13}$$

This equation is satisfied if we insert the proposed $\alpha = (k-1)/k$ and $k = t(n-1)+1$. This result is nongeneric, in the sense that it relies on a special distributional assumption.

[21] Recall that the equilibrium bid function of the FP-APA under our assumptions is $\beta^{\text{FP-APA}}(\theta_i) = \int_0^{\theta_i} g(x)x\,dx$. The corresponding equilibrium payoff is $G(\theta_i)\theta_i - \beta^{\text{FP-APA}}(\theta_i) = \int_0^{\theta_i} G(x)\,dx$.

[22] Consider (11). Under full participation ($\hat{\theta} = 0$), the first term is zero, the second term is the FP-APA-payoff, and the last term is negative. Therefore, the payoff in our auction is lower than under the FP-APA.

Again, for $\alpha = 1$, the first two terms are equal to the expected payoff of the FP-APA, and the last term is an additional payment in the event of losing. In other words, a bidder always pays his or her own bid, but in the event of losing must pay a 'top-up' equal to the difference between the winner's and his or her own bid. As this last term depends on the valuation θ_i, it affects incentives. Therefore, the bid function is different from that of the FP-APA. Compared with the FP-APA, our first auction has the same probability of winning and the same payoff in the event of winning, but the payoff is lower in the case of losing.[23]

An important difference between our auctions and standard all-pay auctions is the role of outside options. For both of our auctions, *full* participation generally requires negative outside options. For example, suppose that valuations are uniformly distributed and recall that full participation requires outside options below $\bar{u} = \Pi(0, 0)$, defined in (9) and (11). These two versions of $\Pi(0, 0)$ are equal to $-(n-1)^2/n$ and $-(n-1)^2/(n^2+n)$, respectively, both of which are clearly negative. Note that the parameters α and k do not appear in the bidders' interim expected equilibrium payoffs, (9) and (11) and, thus, do not affect the participation decision. In terms of comparative statics, as can be seen in (10) and (12), equilibrium bids fully compensate for changes in α and k such that payoffs are unaffected.

Another interesting difference between our auctions and standard all-pay auctions concerns the (un)boundedness of losses. In the FP-APA, as well as in the second-price all-pay auction, bidder i's bid b_i provides a lower bound on i's payoffs: in both auctions, in the event of losing, i's payoff is $-b_i$, while payoffs are higher in the event of winning. Hence, players have a security utility level of zero. In our auctions, by contrast, a bidder's payment is a function of the other players' bids. Thus, losses are potentially large and are not bounded by own bids.

For some of the most well-studied sealed-bid auctions within the standard private-values framework, the literature has shown the uniqueness of the symmetric pure-strategy Bayesian Nash equilibria.[24] These results cover (variants of) the first-price winner-pay and all-pay auction formats, but exclude the Vickrey auction, as well as the second-price all-pay auction (both of which have multiple equilibria). None of the results of which we are aware apply directly to the games we study. However, we have derived a close relationship between our auctions and the first-price all-pay auction. Therefore, we conjecture that equilibrium uniqueness for the games we study may be derived in a way similar to that for existing results. In order intuitively to rule out asymmetric equilibria, consider the nature of the asymmetric equilibria of the Vickrey auction and the second-price all-pay auction. In a nutshell, these equilibria are based on constant bids such that the losers bid low and obtain zero payoffs, whereas the winner makes a high bid, pays little, and cannot improve its payoff by bidding less. Such a situation cannot arise in our games. Consider our first auction format: for given bids of the other players, a loser would always prefer either to not participate or slightly exceed the winner's bid. This is because raising one's bid to slightly above the winner's bid level is almost costless given

[23] For instance, consider the uniform distribution and full participation ($\hat{\theta} = 0$). Then (9) is equal to $-(n - 2 + 1/n) + \theta_i^2/n$, where the first term is negative and the second term is the FP-APA-payoff. The payoff to our auction thus is smaller.

[24] For a brief discussion of the state of the literature, see Chawla and Hartline (2013). Note that a large part of the all-pay auction literature deals with the complete-information case, whereas we are concerned with the standard private-values setting.

that the losers pay (a function of) the winner's bid anyway. The same holds true for the second auction format, but for a different reason. Here, increasing one's bid has a direct proportional cost and might, therefore, not be worthwhile for a loser. However, the designated winner pays a function of all bids, including its own bid. Thus, lowering one's bid improves the payoff, unless there is a tie, i.e., a situation wherein several bidders issue the highest bid. However, this case cannot be part of an equilibrium because, for each one of those bidders, bidding slightly higher and becoming the sole winner would always be preferable.

4 Conclusion

The present study contributes to the analysis of all-pay auctions with payment rules different from the classic first- and second-price rules. We provide results that are complementary to recent studies that cover applications such as litigation or battles with injuries. Our contribution is normative in the sense that we introduce proposal games that, in principle, may find application in a regulatory setting, or, similarly, in situations where a body or group of decision makers can be obliged to propose reductions in their own payoffs. The central idea is that the members of such groups (or industries) may value being known as the proposer of the adopted bill or measure. Unlike the standard contest literature, we interpret bids in our first auction as proposals rather than as requiring costly effort. Only the winning bid becomes payoff relevant.

A policy implication is that, if a reduction of the decision makers' income increases welfare, then transparency, publicity, and a reward for the most aggressive proposal can be utilized to generate Pareto-optimal moves. This holds true even though the decision is a result of independent utility maximization by the participants. Moreover, in a regulatory setting, it may be advantageous if proposals are made by the firms, as they are better informed about their own cost structures and may be willing to forgo short-term profits for an increase in popularity and market share.

We think of the games we study as embedded within larger games. This may be natural in, for example, the case of litigation systems or corporate boardroom votes. Players may be willing to accept negative payoffs in equilibrium because non-participation would imply quitting a larger, presumably overall profitable, game. We take this into account by allowing for arbitrarily large negative or positive outside options.[25]

Our study leaves room for many relevant extensions. Depending on the application, the independent private-values assumption may not be appropriate. Our regulatory proposal game is stylized and could be analyzed within a full-fledged oligopoly game. The issue of collusion may be important. The classes of payment rules we study, as well as the symmetric players assumption, are restrictive. For some applications, a sequential modeling version may be more appropriate. Finally, experimental testing of our results may reveal a better understanding of their applicability.

We have motivated our study with two leading applications, the regulatory proposal game and the all-pay auction with injuries. In addition, we have shown that special cases of our games are relevant within the framework of previous studies of litigation. There is a host of further applications that are

[25] Negative equilibrium payoffs are discussed in, e.g., Krishna and Morgan (1997).

related to the payment rules we study but which fit our model less well. Examples are 'races to the bottom,' (international) tax competition, officials proposing a range of measures after a spending scandal, the social game of 'keeping up with the Joneses,' or the 'filibuster,' in which the politician with the most stamina succeeds and every participant suffers the participants' combined effort costs (e.g., in terms of time or delay, or a loss of public esteem for all politicians). In another group of applications, the payment rule is interpreted as a collective punishment by a third party proportional to (and in retaliation for) attacks on that party, where the attackers are motivated by acclaim for the most daring or successful attack.

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Appendix

Proof of Lemma 1. We need to show that we can rule out symmetric, pure-strategy bid functions that prescribe a strictly positive bid for the lowest participating type $\hat{\theta}$, say $b_0 > 0$ plus a strictly monotone, type-dependent part. In order to see this, consider first the case of full participation. Then, under a monotone bidding function, a bidder with a value between zero and b_0 bids more than b_0 and wins with positive probability. In case of winning, she would have to pay more than her value, while in case of losing her payoff is independent of her own bid. Thus, she can strictly improve her payoff by reducing her bid, contradicting the candidate equilibrium strategy.

Second, consider the case of partial participation and focus on the indifferent bidder with valuation $\theta_i = \hat{\theta}$. With positive probability, the indifferent bidder is the only bidder in the auction. Then she wins with certainty and her payoff is decreasing in her bid. Thus, a smaller bid is better. On the other hand, if there are other active bidders, then, by the definition of being the indifferent bidder, these other bidders have higher values and bid more. Then the indifferent bidder never wins and her payoff is independent of her bid. An incentive to reduce the bid therefore exists. We conclude that every strictly monotone bid function must satisfy $\beta(\hat{\theta}) = 0$. □

Proof of Proposition 1. If $\bar{u} \geq 1$, the outside option is superior to the auction. Thus, we restrict attention to $\bar{u} \in (-\infty, 1)$. We moreover focus on symmetric, pure-strategy equilibria with continuous, strictly monotone bid functions. Therefore, participation is governed by a comparison between the common outside option and the expected equilibrium payoff from participation. This implies a common cutoff valuation, denoted by $\hat{\theta} \in [0, 1]$, below which a player prefers the outside option. This includes the cases of full and no participation. From Lemma 1, we know that $\beta(\hat{\theta}) = 0$.

Suppose that an equilibrium exists wherein all bidders take their outside options if their valuations are below the common threshold value $\hat{\theta}$. If their valuations exceed this value, then they bid according to the strictly monotone bidding function β. We first determine this bidding function, taking the threshold value $\hat{\theta} \in [0,1]$ as given. We start by considering bidder i's decision problem when her value exceeds the threshold value, $\theta_i > \hat{\theta}$. W.l.o.g. we only consider i's positive bids $\beta(z)$ for $z \in (\hat{\theta}, 1]$. Bidder i's expected profit from bidding $\beta(z)$, denoted by $\pi(z, \theta_i)$, is

$$\begin{aligned}
\pi(z,\theta_i) &= G(z)(\theta_i - \alpha\beta(z)) - (1-G(z))\,\mathbb{E}[\alpha\beta(\Theta_{(1:n-1)})|\Theta_{(1:n-1)} > z] \\
&= F^{n-1}(z)(\theta_i - \alpha\beta(z)) - \int_z^1 \alpha\beta(s)dF^{n-1}(s); \\
\frac{\partial \pi(z,\theta_i)}{\partial z} &= (F^{n-1}(z))'(\theta_i - \alpha\beta(z)) - F^{n-1}(z)\alpha\beta'(z)) + \alpha\beta(z)(F^{n-1}(z))' = 0 \\
\iff \quad \beta'(z) &= \frac{1}{\alpha}\frac{(F^{n-1}(z))'}{F^{n-1}(z)}\theta_i = \frac{(n-1)}{\alpha}\frac{F(z)^{n-2}(z)f(z)}{F^{n-1}(z)}\theta_i = \frac{(n-1)}{\alpha}\frac{f(z)}{F(z)}\theta_i.
\end{aligned} \tag{16}$$

As argued above, $\beta(\hat{\theta}) = 0$. Using this as the boundary condition, our equilibrium bid function is found by integrating the above differential equation, with one exception: Because $F(0) = 0$, the differential equation is not defined at $\hat{\theta} = 0$. In that case, we integrate over the open interval $(0, \theta_i]$ and define $\beta(0) = 0$, corresponding to $\beta(\hat{\theta}) = 0$. In symmetric equilibrium, $z = \theta_i$. Thus,

$$\beta(\theta_i) = \frac{(n-1)}{\alpha}\int_{\hat{\theta}}^{\theta_i} \frac{f(x)x}{F(x)}\,dx. \tag{17}$$

This corresponds to (10). Note that, at $x = 0$, the numerator as well as the denominator of the integrand vanish. Thus, the integrand is bounded by l'Hôspital's rule,

$$\lim_{x\to 0}\frac{f(x)x}{F(x)} = \lim_{x\to 0}\frac{f'(x)x}{f(x)} + 1 = 1, \tag{18}$$

and, therefore, the integrand of (17) is bounded everywhere. Since every bounded function can be integrated, the resulting bidding function is well defined.

We now turn to i's expected equilibrium profit, denoted by $\Pi(\hat{\theta}, \theta_i)$, where all players follow (17) and $\theta_i > \hat{\theta}$.

$$\begin{aligned}
\Pi(\hat{\theta},\theta_i) &= F^{n-1}(\theta_i)(\theta_i - \alpha\beta(\theta_i)) - \int_{\theta_i}^1 \alpha\beta(x)\,dF^{n-1}(x) \\
&= F^{n-1}(\theta_i)\left(\theta_i - (n-1)\int_{\hat{\theta}}^{\theta_i}\frac{f(y)y}{F(y)}\,dy\right) - \underbrace{\int_{\theta_i}^1 (n-1)\int_{\hat{\theta}}^x \frac{f(y)y}{F(y)}\,dy(F^{n-1}(x))'\,dx}_{=:(A)}.
\end{aligned} \tag{19}$$

Interchanging the order of integration, the last term can be written as

$$\begin{aligned}
(A) &= \int_{\hat{\theta}}^{\theta_i}\int_{\theta_i}^1 (F^{n-1}(x))'\,dx(n-1)\frac{f(y)y}{F(y)}\,dy + \int_{\theta_i}^1\int_y^1 (F^{n-1}(x))'\,dx(n-1)\frac{f(y)y}{F(y)}\,dy \\
&= \int_{\hat{\theta}}^{\theta_i}(1-F^{n-1}(\theta_i))(n-1)\frac{f(y)y}{F(y)}\,dy + \int_{\theta_i}^1 (1-F^{n-1}(y))(n-1)\frac{f(y)y}{F(y)}\,dy.
\end{aligned} \tag{20}$$

After reinserting (A) into (19) and straightforward simplification, we obtain

$$\Pi(\hat{\theta},\theta_i) = F^{n-1}(\theta_i)\theta_i - (n-1)\int_{\hat{\theta}}^1 \frac{f(x)x}{F(x)}\,dx + \int_{\theta_i}^1 \underbrace{F^{n-1}(x)(n-1)\frac{f(x)x}{F(x)}}_{=(F^{n-1}(x))'x}\,dx. \tag{21}$$

Integrating the last term by parts, we get

$$\begin{aligned}\Pi(\hat{\theta},\theta_i) &= F^{n-1}(\theta_i)\theta_i - (n-1)\int_{\hat{\theta}}^{1}\frac{f(x)x}{F(x)}dt + 1 - F^{n-1}(\theta_i)\theta_i - \int_{\theta_i}^{1}F^{n-1}(x)\,dx \\ &= 1-(n-1)\int_{\hat{\theta}}^{1}\frac{f(x)x}{F(x)}\,dx - \int_{\theta_i}^{1}F^{n-1}(x)\,dx,\end{aligned} \tag{22}$$

which equals $\Pi(\hat{\theta},\theta_i)$ in (9). Given this participation payoff, we determine participation behavior by taking into account the commonly known outside options. Since $\Pi(\hat{\theta},\theta_i)$ is obviously strictly increasing in both, θ_i and $\hat{\theta}$, the minimum of (9) is $\Pi(0,0)$ and the maximum is $\Pi(1,1) = 1$. Therefore, we have full participation for $\bar{u} \leq \Pi(0,0)$, no participation for $\bar{u} \geq \Pi(1,1) = 1$, and participation according to the cutoff value $\hat{\theta} \in (0,1)$ for $\Pi(0,0) < \bar{u} < \Pi(1,1) = 1$. In the latter case, the cutoff is defined by $\bar{u} = \Pi(\hat{\theta},\hat{\theta})$, the expected payoff of the indifferent bidder.

It remains to be shown that (17) does not only satisfy the first-order condition of i's best response problem but that it is indeed a maximizer for $\theta_i > \hat{\theta}$.

Similar to the above, consider $\theta, z \in (\hat{\theta}, 1]$. Start from (16) and insert (17), cancelling out α:

$$\pi(z,\theta) = F^{n-1}(z)\left(\theta - (n-1)\int_{\hat{\theta}}^{z}\frac{f(x)x}{F(x)}dx\right) - \underbrace{\int_{z}^{1}(n-1)\int_{\hat{\theta}}^{y}\frac{f(x)x}{F(x)}\,dx(F^{n-1}(y))'\,dy}_{=:(A)}. \tag{23}$$

Interchanging the order of integration, the term (A) can be written as

$$(A) = (n-1)\Bigg(\int_{\hat{\theta}}^{z}\frac{f(x)x}{F(x)}\underbrace{\int_{z}^{1}(F^{n-1}(y))'\,dy}_{=1-F^{n-1}(z)}\,dx + \int_{z}^{1}\frac{f(x)x}{F(x)}\underbrace{\int_{x}^{1}(F^{n-1}(y))'\,dy}_{=1-F^{n-1}(x)}\,dx\Bigg). \tag{24}$$

Now, reinsert into (23), cancel out the terms $\pm\, F^{n-1}(z)(n-1)\int_{\hat{\theta}}^{z}\frac{f(x)x}{F(x)}dx$, and get

$$\begin{aligned}\pi(z,\theta) &= \underbrace{F^{n-1}(z)\theta}_{=:(B)} - (n-1)\left(\int_{\hat{\theta}}^{z}\frac{f(x)x}{F(x)}\,dx + \int_{z}^{1}\frac{f(x)x}{F(x)}(1-F^{n-1}(x))\,dx\right) \\ &= \underbrace{\theta - \int_{z}^{1}(F^{n-1}(x))'\theta\,dx}_{=(B)} - (n-1)\int_{\hat{\theta}}^{1}\frac{f(x)x}{F(x)}\,dx + \underbrace{(n-1)\int_{z}^{1}\frac{f(x)x}{F(x)}F^{n-1}(x))\,dx}_{\int_z^1(F^{n-1}(x))'x\,dx} \\ &= \theta - (n-1)\int_{\hat{\theta}}^{1}\frac{f(x)x}{F(x)}\,dx + \underbrace{\int_{z}^{1}(F^{n-1}(x))'(x-\theta)\,dx}_{=:(C)}.\end{aligned} \tag{25}$$

Note, that only (C) depends on z. Thus,

$$\pi(\theta,\theta) - \pi(z,\theta) = \int_{\theta}^{z}(F^{n-1}(x))'(x-\theta)\,dx. \tag{26}$$

Since this is strictly positive for all $z \neq \theta$, where $z, \theta \in [0,1]$, this completes the proof. □

Proof of Proposition 2. Obviously, if $\bar{u} \geq 1$, the outside option is superior to participation. Thus, we only need to consider $\bar{u} \in (-\infty, 1)$. We restrict attention to symmetric, pure-strategy equilibria with continuous, monotone bid functions. Therefore, participation is governed by a comparison of (candidate) equilibrium payoffs from participation, and the outside option. This implies a common cutoff valuation, denoted by $\hat{\theta} \in [0,1]$, above which a player participates in the auction. This includes the cases of full and no participation. Similar to the proof of Lemma 1, we can show that $\beta(\hat{\theta}) = 0$. The only difference is that, here, losing bidders' payoff is decreasing in (rather than independent of) the own bid.

Suppose that there is an equilibrium wherein player i takes the outside option if $\theta_i < \hat{\theta}$ and otherwise participates and follows the common bid function β. Obviously, (12) is strictly increasing in θ_i. Suppose that player i's rivals j participate if $\theta_j \geq \hat{\theta}$ and, if so, bid according to (12). W.l.o.g. we only consider bidder i's deviating bids $\beta(z)$ where $z \in [\hat{\theta}, 1]$ (and discuss outside options below). Recalling that $G = F^{n-1}$, and denoting a valuation by the random variable Θ, bidder i's expected payoff is

$$\begin{aligned} \pi(z,\theta) &= F^{n-1}(z)\theta - \alpha\beta^k(z) - (n-1)\,\mathbb{E}[\alpha\beta(\Theta)^k], \\ \frac{\partial \pi(z,\theta)}{\partial z} &= (F^{n-1}(z))'\theta - (\alpha\beta^k(z))'. \end{aligned} \tag{27}$$

Applying the symmetric equilibrium condition, $z = \theta$, and the boundary condition $\beta(\hat{\theta}) = 0$ (see above), integration delivers (12). As a proof of sufficiency, we show that $\pi_i(\theta,\theta) > \pi_i(z,\theta)$ for all $z \in [\hat{\theta}, 1]$, $z \neq \theta$. Note that the term $-\alpha(n-1)\,\mathbb{E}[\beta(\Theta)^k]$ cancels out in the first line of (27) when we compute the difference $\pi_i(\theta,\theta) - \pi_i(z,\theta)$. Inserting the candidate payoff, we get

$$\begin{aligned} \pi(\theta,\theta) - \pi(z,\theta) &= \theta(F^{n-1}(\theta) - F^{n-1}(z)) - \int_z^\theta \left(F^{n-1}(x)\right)' x\,dx \\ &= \theta \int_z^\theta \left(F^{n-1}(x)\right)'\,dx - \int_z^\theta \left(F^{n-1}(x)\right)' x\,dx \\ &= \int_z^\theta \left(F^{n-1}(x)\right)'(\theta - x)\,dx > 0. \end{aligned} \tag{28}$$

We continue by computing a participating bidder i's (candidate) expected equilibrium payoff, denoted by $\Pi(\hat{\theta}, \theta_i)$, i.e., we take the cutoff value $\hat{\theta}$ as given, and assume $\theta_i \geq \hat{\theta}$ and consider participating rivals' bids for $\theta_j \geq \hat{\theta}$, $j \neq i$. Inserting the candidate payoff into the first line of (27), we get

$$\Pi(\hat{\theta}, \theta_i) = \underbrace{F^{n-1}(\theta_i)\theta_i - \int_{\hat{\theta}}^{\theta_i} (F^{n-1}(x))' x\,dx}_{=:(A)} - (n-1) \underbrace{\int_0^1 \int_{\hat{\theta}}^{\theta_j \geq \hat{\theta}} (F^{n-1}(x))' x\,dx f(\theta_j) d\theta_j}_{=:(B)}. \tag{29}$$

Using integration by parts, term (A) in the above can be written as

$$\begin{aligned} (A) &= F^{n-1}(\theta_i)\theta_i - \left(\left[F^{n-1}(x)x\right]_{\hat{\theta}}^{\theta_i} - \int_{\hat{\theta}}^{\theta_i} F^{n-1}(x)\,dx \right) \\ &= F^{n-1}(\hat{\theta})\hat{\theta} + \int_{\hat{\theta}}^{\theta_i} F^{n-1}(x)\,dx. \end{aligned} \tag{30}$$

Interchanging the order of integration, term (B) can be written as

$$(B) = \int_{\hat{\theta}}^{1} \int_{x}^{1} f(\theta_j) d\theta_j (F^{n-1}(x))' x \, dx = \int_{\hat{\theta}}^{1} (1 - F(x))(F^{n-1}(x))' x \, dx. \tag{31}$$

Reinserting these expressions for (A) and (B) gives (11).

Finally, given this participation payoff, we determine participation behavior by taking into account the value of the commonly known outside options. Note that (11) is strictly increasing in both θ_i and $\hat{\theta}$. The former is obvious and the latter can be seen in the first derivative,

$$\frac{\partial \Pi(\hat{\theta}, \theta_i)}{\partial \hat{\theta}} = (F^{n-1}(\hat{\theta}))'\hat{\theta} + F^{n-1}(\hat{\theta}) - F^{n-1}(\hat{\theta}) + (n-1)(1 - F(\hat{\theta}))(F^{n-1}(\hat{\theta}))'\hat{\theta}, \tag{32}$$

where the second and third term cancel out, while the remaining terms are positive. Thus, for $\hat{\theta}, \theta_i \in [0, 1]$ and $\theta_i \geq \hat{\theta}$, the maximum of (11) is $\Pi(1, 1) = 1$ while the minimum is $\Pi(0, 0)$. Therefore, we have full participation for $\bar{u} \leq \Pi(0, 0)$, and participation according to the cutoff value $\hat{\theta} \in (0, 1)$ whenever $\Pi(0, 0) < \bar{u} < \Pi(1, 1) = 1$. In the latter case, the cutoff is defined by $\bar{u} = \Pi(\hat{\theta}, \hat{\theta})$, the payoff of the indifferent bidder. □

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