# Optimal Design of Angular Position Sensors

Hubert Zangl*, Lisa-Marie Faller*and Wolfgang Granig†

*Institute of Smart System Technologies, Universitaetstrasse 65-67, A-9020 Klagenfurt
† Infineon Technologies AG, Villach, Austria
E-mail: hubert.zangl@aau.at

***Abstract*—A variety of angular position sensors make use of field patterns that are rotated together with a rotor with respect to a stator. The determination of the relative angular displacement of the rotor is then determined using at least two e.g. magnetic field sensors that spatially sample from the field pattern. In the sensor model we include spatial dependencies of random deviations by means of a Gaussian random field. Based on this, an approach for fast determination of D-optimal designs is presented. These results show that the commonly used distributions of magnetic field sensors are actually optimal for the determination of inphase and quadature signals in the presence of spatial correlations, provided that the number of field sensors is higher or equal to three. However, in the uncorrelated case, the common solutions are not the only optimal solutions. Furthermore, it is shown by means of numerical studies that the optimal design with respect to the determination of the rotational angle may deviate from the common solution depending on the nature of the random field. It is found that a restriction to symmetric designs is not necessary. Thus, the design space can be extended to allow for further improvements of such angular position sensors.**

***Index Terms*—D-optimal Design, angular position sensor**

## I. Introduction

Non-contact measurement of angular positions offers advantages such as low wear, low acoustic noise generation, insensitivity to vibrations and contaminations, etc. Due to the intrinsic robustness, non-contact angular positions sensors based on magnetic sensors are very popular. In such sensors a field pattern, which is typically generated using a permanent magnet, is rotated together with a shaft in order to determine the relative angular displacement between the rotor/shaft and the stator. The sensor output is calculated from measurements obtained with magnetic field sensors. Commonly, sensors such as anisotropic magnetoresistive sensors (AMR, e.g. [1]), giant magnetoresistive sensors (GMR, e.g. [2]), tunneling magnetoresistive sensors (TMR, e.g. [3]) and Hall effect sensors (e.g. [4]) are used. Illustrations of the according principles are shown in Fig. 1. Similar approaches are also found in non-magnetic sensors, e.g. in capacitive and optical sensors. Frequently, the magnetic field observed at the locations of the magnetic field sensors shows an almost sinusoidal variation over the relative angular rotation of the magnet. The magnetic field sensors are usually placed in such a way, that the obtained signals are phase shifted by $90^o$, such that an inphase and a quadrature signal can be determined. This approach is illustrated in Fig. 2. In this paper we investigate the optimal choice of the sensor design, i.e. if the commonly used choice of the location and/or the orientation of the magnetic field sensors is optimal or if better solutions can be found.

## II. Sensor Model and Optimization Parameters

A simple measurement model for the relation between the magnetic field measured with a magnetic field sensor and the relative rotation $\theta$ of a magnet is given by

$$S(\theta) = A\, sin(\theta - \varphi) + w \quad (1)$$

where $S$ is the magnetic field sensor output signal, $\theta$ the relative angular position, $A$ the amplitude, and $\varphi$ a phase shift linked to the sensor location/orientation (compare [2]). The term $w$ summarizes random deviations and noise. Typically, both the amplitude and the phase shift have to be considered as unknown. Thus, at least two magnetic field sensor are required. Typcially, the actual number of field sensors is even higher in order to compensate non-ideal effect. Thus, a measurement vector $\mathbf{S}(\theta, \boldsymbol{\eta})$, representing the measurement as obtained from several sensors located on the integrated circuit, is obtained.

$$\mathbf{S}(\theta, \boldsymbol{\eta}) = A(\boldsymbol{\eta})\, sin(\theta - \boldsymbol{\varphi}(\boldsymbol{\eta})) + \mathbf{w} \quad (2)$$

The design parameter vector $\boldsymbol{\eta}$ comprises all information describing the configuration, such as the magnetic sensor location on the silicon die, the orientation of the magnetization of a fixed layer, etc. The vector $\mathbf{w}$ represents the random deviations from the ideal model. The contributions to this vector are from random measurement noise but also from other sources, as will be discussed in section IV.

In classical optimal design (e.g. [5], [6]), it is aimed to find the configuration $\boldsymbol{\eta}_{opt}$ that maximizes the Fisher Information $I$ of the configuration. The inverse of the Fisher Information $I$ represents a lower bound for the

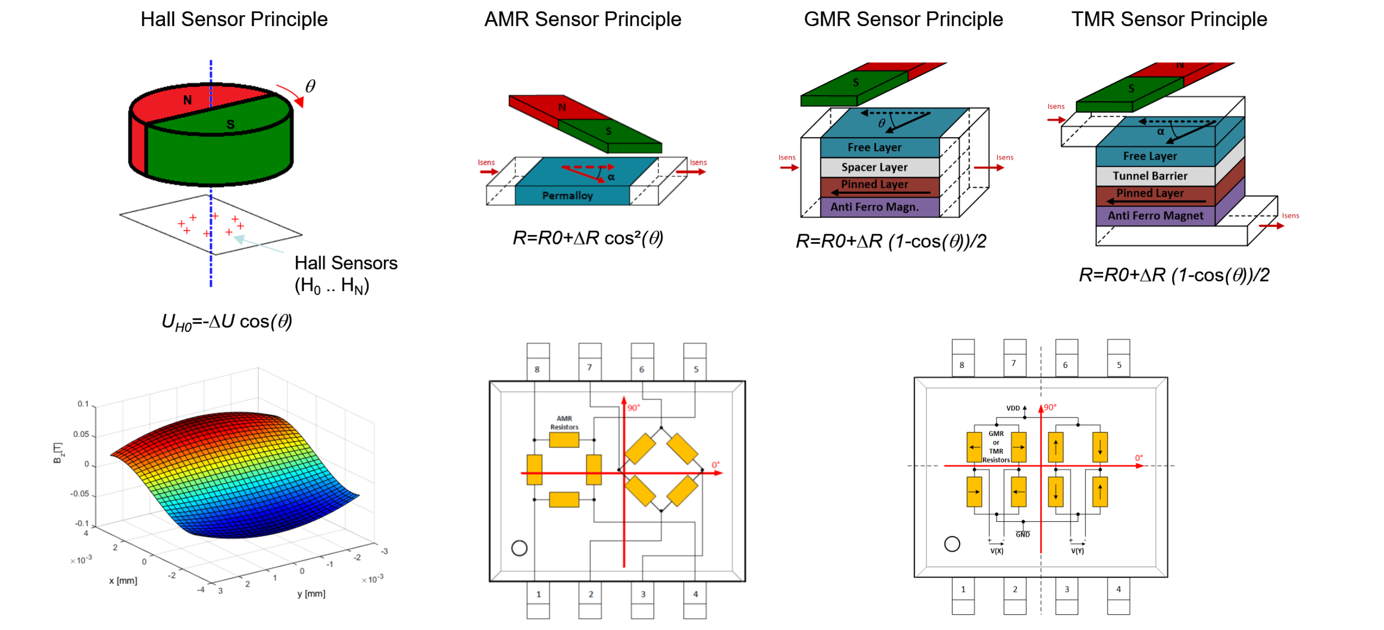


Fig. 1. Examples for magentic angular position sensors. In Hall sensors the magnetic field strength translates into a proportional voltage signal. When the magnet and thus the field is rotated, an approximately sinusoidal variation is observed near the axis of rotation. In magnetoresisitive sensors such as AMR, GMR and TMS the sensors resistance depends on the angle between a free layer and a reference orientation. As the free layer follows the rotation of the strong external magnet, the resistance changes with the rotation. Bridge circuits are then used to obtain corresponding voltage signals, as shown in Fig. 2. While the periodicity for AMR sensors is $180^o$, the periodicity of the other sensors is $360^o$.

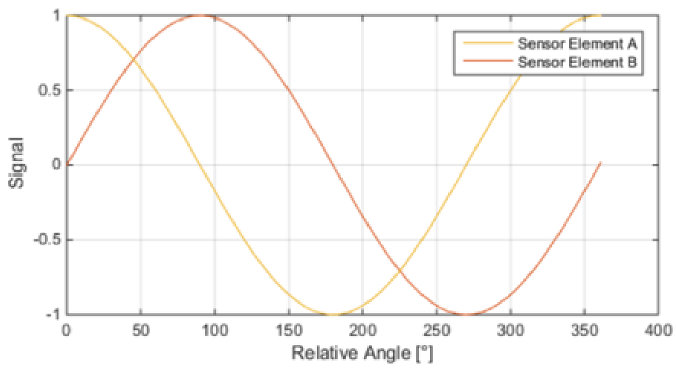


Fig. 2. Example signals as obtained for a rotation of a magnet for different sensor elements. Depending on e.g. the location of the sensors or the orientation of a pinned layer fixed layer, the signal gets phase shifted. Having two linearily independet signals, the angle can be determined even when the amplitude of the signals is unknown.

variance of any unbiased estimator for the parameter in question, i.e.

$$var(\hat{\theta}) \geq \frac{1}{I(\theta, \boldsymbol{\eta})} \tag{3}$$

with the Fisher Information given by

$$I(\theta, \boldsymbol{\eta}) = -E\left[\frac{\partial^2 \ln p(x; \theta, \boldsymbol{\eta})}{\partial \theta^2}\right] \tag{4}$$

and the optimal design

$$\boldsymbol{\eta}_{opt} = \arg\max_{\boldsymbol{\eta} \in H} I(\theta, \boldsymbol{\eta}) \tag{5}$$

where $H$ represents the design space. If the uncertainty of the measurements $w$ can be modeled using an jointly Gaussian distribution

$$W \sim \mathcal{N}(0, \mathbf{C}) \tag{6}$$

where $\mathbf{C}$ represents the covariance matrix of the (zero mean) random deviation vector, then the Fisher Information can be obtained from

$$[\boldsymbol{I}(\boldsymbol{\theta}, \boldsymbol{\eta})]_{ij} = \left[\frac{\partial \mathbf{h}(\boldsymbol{\theta}, \boldsymbol{\eta})}{\partial \theta_i}\right]^T \boldsymbol{C}^{-1}(\boldsymbol{\theta}, \boldsymbol{\eta}) \left[\frac{\partial \mathbf{h}(\boldsymbol{\theta}, \boldsymbol{\eta})}{\partial \theta_j}\right] + \frac{1}{2}\mathrm{tr}\left[\boldsymbol{C}^{-1}(\boldsymbol{\theta}, \boldsymbol{\eta}) \frac{\partial \boldsymbol{C}(\boldsymbol{\theta}, \boldsymbol{\eta})}{\partial \theta_i} \boldsymbol{C}^{-1}(\boldsymbol{\theta}, \boldsymbol{\eta}) \frac{\partial \boldsymbol{C}(\boldsymbol{\theta}, \boldsymbol{\eta})}{\partial \theta_j}\right] \tag{7}$$

In the given problem the approach has two drawbacks: First, as (3) is just a lower bound there is no guarantee that an algorithm exists that actually gets close to it, i.e. is not guaranteed that the estimator variance of the so found design can actually attain the lowest possible value. This is particularly relevant as signal processing on small integrated sensor devices typically has quite limited computational power. Secondly, the Fisher Information depends on the unknown angular position. The classical approach to use locally optimal designs is not applicable as the value for the angle can vary over a wide range. Another approach to overcome the second issue is to use Bayesian optimal design, where the expectation of a utility function $U$ is maximized [7],

$$\boldsymbol{\eta}_{opt} = \arg\max_{\boldsymbol{\eta} \in H} \int_S \int_\theta U(\boldsymbol{\eta}, \theta, \mathbf{S}) p(\theta, \mathbf{S}|\boldsymbol{\eta}) d\theta d\mathbf{S} \tag{8}$$

where $p(\theta, \mathbf{S}|\boldsymbol{\eta})$ denotes the joint probability density function for the observed data $\mathbf{S}$ and the parameter $\theta$ for a given design $\boldsymbol{\eta}$. The utility function is usually based on the posterior probability density $p(\theta|\mathbf{S}, \boldsymbol{\eta})$.

For sensors, a natural choice for the utility function is the mean square error of the estimator, such that the expectation for the mean square error would be minimized. However, in order to determine the mean square error, an estimator needs to be known already in the design. If we restrict the estimators to be of simple nature, e.g. linear in the data, the Bayesian Minimum Mean Square Error estimator (BMMSE) could be used. Even though the estimator itself is simple (of low computational cost), the determination of it is not, as can be seen e.g. in [8] where this approach is applied to fast reconstruction in Electrical Capacitance Tomography. Considering that we would have to perform this approach for every candidate design, the computation effort gets quite high. Furthermore, the linear estimators are not applicable to model (2) due to the substantial non-linearity. Thus, we could only use piecewise linearization, which further increases the complexity.

A variety of other utility functions have been developed e.g. using the Kullback-Leibler divergence, for which

$$U_1(\boldsymbol{\eta}) = \int_S \int_\theta \log p(\theta|\mathbf{S}, \boldsymbol{\eta}) p(\mathbf{S}, \theta|\boldsymbol{\eta}) d\theta d\mathbf{S} \quad (9)$$

is maximized. A recent review on Bayesian methods is provided in [7]. Bayesian design has mostly been limited to simple models (e.g. low dimensional linear and nonlinear fixed effects models) because of the computational challenges of performing the integration of (8) or (9) and maximization thereof; and the use of standard optimization algorithms, such as the NewtonRaphson method, to find the optimal design may be inappropriate. Even though novel computational strategies to speed up the process to solve Bayesian optimal design problems have been suggested, they still use comparatively expensive methods such as Markov Chain Monte Carlo and Sequential Monte Carlo methods [9].

Besides the Bayesian approach, a maximin approach (e.g. [10]) may be used, i.e. maximizing the minimum of the determinant of the information matrix over all possible values for the parameter $\theta$.

$$\boldsymbol{\eta}_{opt} = \arg \max_{\boldsymbol{\eta} \in H} \min_{\theta \in \Theta} I(\theta, \boldsymbol{\eta}) \quad (10)$$

However, the problem remains that there is no guarantee that the lower bound can be attained.

For angular position sensors we suggest a different approach based on the assumption that an accurate determination of the inphase and quadrature (I/Q) contributions will allow for an accurate estimation of the angular position $\theta$. As will be demonstrated, in this case it is possible to actually attain the lower bound - yet for the $I$ and $Q$ contributrions but not directly for the angle $\theta$. Additionally, the computational complexity of the approach is quite low.

## III. Optimal Design for Inphase and Quadrature Contributions

The model (2) may be rewritten as

$$\begin{aligned} \mathbf{S}(\theta, \boldsymbol{\eta}) &= I_A(\theta, \boldsymbol{\eta}) \cdot cos(\varphi(\boldsymbol{\eta})) + \quad (11) \\ &\quad + Q_A(\theta, \boldsymbol{\eta}) \cdot sin(\varphi(\boldsymbol{\eta})) + \mathbf{w} \\ &= \mathbf{H}(\boldsymbol{\eta}) \begin{bmatrix} I_A \\ Q_A \end{bmatrix} + \mathbf{w} = \mathbf{H}(\boldsymbol{\eta})\boldsymbol{\theta} + \mathbf{w} \quad (12) \end{aligned}$$

where $I_A$ and $Q_A$ represent the inphase and quadrature contributions of the observed measurement vector $\mathbf{S}$. Please note that the individual magnetic fields sensors may observe individual amplitudes. However, this may be corrected by individual gain factors, such that common values for $I_A$ and $Q_A$ can be used as given in (11). The values for $I_A$ and $Q_A$ are summarized in the vector ($\boldsymbol{\theta}$).

Once we have estimate the $I_A$ and $Q_A$ contributions $\boldsymbol{\theta}$, the estimate for the angle $\theta$ is obtained using

$$\theta = \operatorname{atan2}(Q_A, I_A) = \operatorname{atan2}(\theta_2, \theta_1) \quad (13)$$

where atan2 is arctangent function with two arguments, i.e. the arctangent function including the appropriate assignment of the quadrant of the computed angle.

In contrast to a direct optimization for the angle $\theta$, we now have to determine a vector of unknowns. Consequently, the Fisher Information becomes a matrix. For this class of problems, however, the equality condition holds:

$$var(\hat{\theta}_i) = \left[\mathbf{I}^{-1}\right]_{ii} \quad (14)$$

with

$$\hat{\boldsymbol{\theta}} = (\mathbf{H}^T \mathbf{C}^{-1} \mathbf{H})^{-1} \mathbf{H}^T \mathbf{C}^{-1} \mathbf{S} \quad (15)$$

When the covariance $\mathbf{C}$ does not depend on $\theta$, the Fisher Information

$$\mathbf{I} = (\mathbf{H}^T \mathbf{C}^{-1} \mathbf{H}) \quad (16)$$

does also not depend on $\boldsymbol{\theta}$, as $\mathbf{H}$ does not depend on $\boldsymbol{\theta}$. Therefore, no maximization of averaging over $\theta$ is needed, and a solution will be optimal (in the chosen sense) for all $\boldsymbol{\theta}$.

However, as we now estimate two parameters, a design will in general not be optimal for the determination of both parameters within $\boldsymbol{\theta}$. A balancing between the parameters is needed. Common approaches are to look at the determinant of $\mathbf{I}$ (D-optimality) or the trace of $\mathbf{K}\mathbf{I}^{-1}$ (A-optimality). The first approach bears the risk that the estimates may have strongly correlated errors. The second approach requires a weighing matrix $\mathbf{K}$, if the parametes are of different nature. As this is not the case in the given problem, $\mathbf{K}$ can be choosen as identity matrix. However, for the experimental results reported in this paper, only D-optimality was used and the objective function $\Gamma$ is given by

$$\Gamma(\boldsymbol{\eta}) = \left|I(\boldsymbol{\eta})^{-1}\right| \tag{17}$$

as our objective function and the optimal design is given by

$$\boldsymbol{\eta}_{opt} = \arg\min_{\boldsymbol{\eta}\in H} \Gamma(\boldsymbol{\eta}) \tag{18}$$

The optimal design can be found using numeric optimization, e.g Nelder-Mead method [11]. Random deviations as described in section IV are modeled and summarized using random fields, which allows for an optimization that is based on a comparatively simple model and thus for a fast determination of D-optimal designs.

## IV. Modeling of Random Deviations

Looking at (7) we see that we need to know the sensitivity of our magentic field sensors with respect to the parameter of interest but also the covariance matrix of the measurements given by $\mathbf{C}$. Possible contributions to $\mathbf{C}$ are:

- Magnet alignment
  - Axial displacement
  - Radial displacement
  - Inclination
- Magnet Magnetization Defects
- Disturbers
  - External fields
  - Ferromagnetic materials in the vicinity
- Sensor elements
  - Measurement noise
  - Offset, gain errors, e.g. due to process variations
  - Piezo-resistive effect
  - Piezo-Hall Effect

Except for measurement noise the sources of uncertainty will typically affect not just a single sensor element but lead to correlated effects on the entire sensor array.

For example, piezoresistive [12] and the piezo-Hall effect [13] are related to mechanical stress. The electric field $\vec{E}$ is given by

$$\vec{E} = \breve{\rho}\vec{J} - \left(\breve{K}\vec{B}\right) \times \vec{J} \tag{19}$$

where $\breve{\rho}$ represents the resistivity tensor, $\vec{J}$ the current density, $\vec{B}$ the magnetic flux density, and $\breve{K}$ the Hall coefficients tensor. This relation gets disturbed by deviations of the resistivity $\rho$ due to the piezoresistive effect

$$\delta\breve{\rho} = \frac{\Delta\breve{\rho}}{\rho_0} = \breve{\Pi}\breve{\sigma} \tag{20}$$

where $\breve{\Pi}$ and represents the tensor of piezoresisitve coefficients; and

$$\delta\breve{K} = \frac{\Delta\breve{K}}{K_0} = \breve{P}\breve{\sigma} \tag{21}$$

where $\breve{\Pi}$ represents the tensor of the piezo-Hall coefficients; $\rho_0$ and $K_0$ are the scalar resistivity and Hall coefficient for the unstrained case [14]. As stress can have significant influence on the performance of integrated sensors [15], it is important to be considered.

Consequently, the distribution of stress will lead to a corresponding distribution of stress induced deviations from the nominal behavior. The correlation can be determined experimentally (e.g. [14]), but also using Finite Element Method (FEM) simulation. Fig. 3 shows an example. Clearly, points that are in vicinity to each othe experience similar displacement and similar stress.

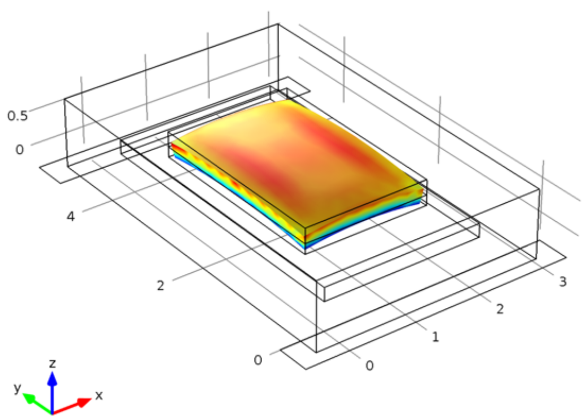


Fig. 3. Simplified FEM Model of an integrated circuit within a package. Due to different thermal properties (but also due to process related effects), substantial deformation and stress is present across the circuit. However, the stress at close locations is similar.

As a consequence stress induced deviations are subject to spatial correlations. Fig. 4 illustrates the impact of temperature variation on the stress over a integrated sensor. If the temperature is unknown, it might be treated as a random variable. With this approach, also the stress distribution and finally the deviations due to piezoresistive and piezo-Hall effect become random variables. However, as the stress at different locations will not be independent, the deviations of the measurement signals will also not be independent. Therefore, this must be considered in the covariance matrix of the measurements. We suggest to use a Gaussian random field to model this relation. The covariance matrix $\mathbf{C}$ is then a function of the design parameters

$$C_{i,j} = \mathrm{cov}(w_i, w_j) = f(\boldsymbol{\eta_i}, \boldsymbol{\eta_j}) \tag{22}$$

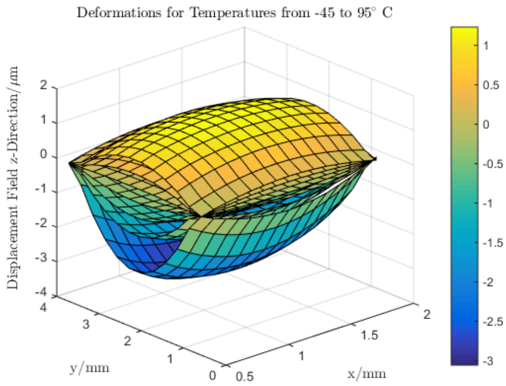


Fig. 4. Temperature dependent deformation of a silicon die (compare Fig. 5). If we treat the unknown temperature as a random variable, also the stress deviation becomes a random variable. The stress changefor points in close vicinity to each other will be similar.

where $\boldsymbol{\eta_i}$ represents the part of the parameter vector $\boldsymbol{\eta}$ that correspond to the $i^{th}$ sensor element; e.g. for a Hall sensor this $\boldsymbol{\eta_i}$ may be the cartesian coordinates of the sensor locations. If the random field is stationary and isotropic, then the covariance for different locations only depends on the Euclidian distance between the parameter vectors,

$$C_{i,j} = \text{cov}(w_1, w_2) = f(||\boldsymbol{\eta_i} - \boldsymbol{\eta_j}||) \quad (23)$$

In case of Hall sensors, this could be

$$C_{i,j} = \text{cov}(w_1, w_2) = f(||\vec{x}_i - \vec{x}_j||) \quad (24)$$

where $\vec{x}_i$ represents the coordinates of the $i^{th}$ sensor within the silicon die.

A common choice for the covariance function between two observations made with two sensors is the squared exponential

$$C_{ij} = \sigma_f^2 e^{-\frac{||\boldsymbol{\eta_i} - \boldsymbol{\eta_j}||^2}{2l^2}} \quad (25)$$

$\sigma_f^2$ is the variance of a measurement and $l$ is a length parameter that is found based on simulation results or experimental evaluation according to [16].

It may also be useful to consider model errors by an appropriate choice of $\mathbf{C}$. For example, a metamodel may be used to reduce quantization noise in a FEM model. However, the errors introduced by the metamodel will also be similar for points in close vicinity to each other, as shown in Fig. 5. Even though the deviations are systematic, it may be useful to consider this in the choice of $l$ in model (25)

## V. Optimization Approach and Results

In general, analytic solutions for (18) are hard to find. Therefore, except for simple cases, we use a numeric optimization approach. For the results shown in this paper

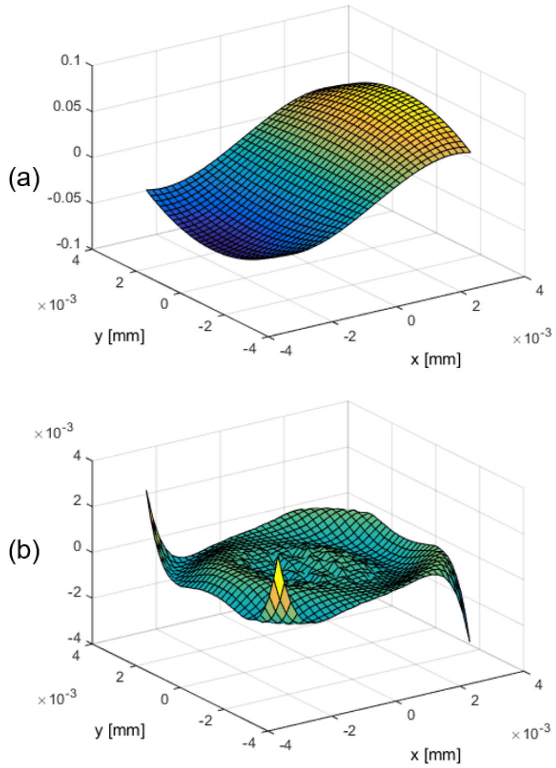


Fig. 5. Example of using a metamodel to reduce quantization noise. If the field is obtained using FEM simulations (e.g. due to the presence of ferromagnetic material in the environment), quantization noise is introduced. This can be estimated using the deviation with respect to a smoothed model, e.g. a polynomial metamodel. However, this will also introduce deviations. Within the vicinity around a certain location, the deviation will be similar.

we use the Nelder-Mead algorithm [11] with randomized start vectors in order to find the optimal design. An analytic solution for the placement of two magnetic sensors and uncorrelated random deviations if provided in the Appendix.

Different scenarios have been studied. Fig. 6 shows solutions for the uncorrelated case, i.e. $\mathbf{C}$ as identity matrix. For two sensor elements, a phase shift of $90^o$ is obtained, for three sensors a phase shift of $60^o$ as expected. However, for four sensors, a phase shift of $90^o$ between all magnetic field sensors is not a unique optimum solution, any solution where two sensors in pairs are separated by $90^o$ is also optimal.

If we look at the objective function it turns out that there is not a unique solution even in the case of three sensors (compare Fig. 7). However, if we introduce a random common offset to the sensor elements, the objective function changes significantly. Only one solution with a phase shift of $120^o$ remains (please note the exchangeability of the individual sensors, therefore the objective function has two peaks, but the solutions are equivalent).

When we increase the number of sensors it turns out

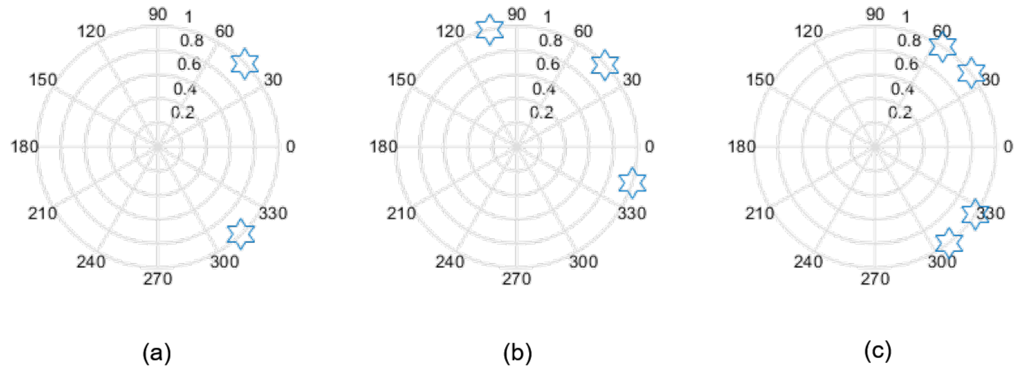


Fig. 6. Example solutions for the uncorrelated case. (a) Two sensors. The optimal solution is found for a phase difference of $90^o$. (b) Three sensors. Besides the phase difference of $120^o$, also a phase difference of $60^o$ is optimal. (c) Four sensors. Two pairs with a phase shift of $90^o$ are found as optimal solution. However, the phase shift between the pairs is arbitrary.

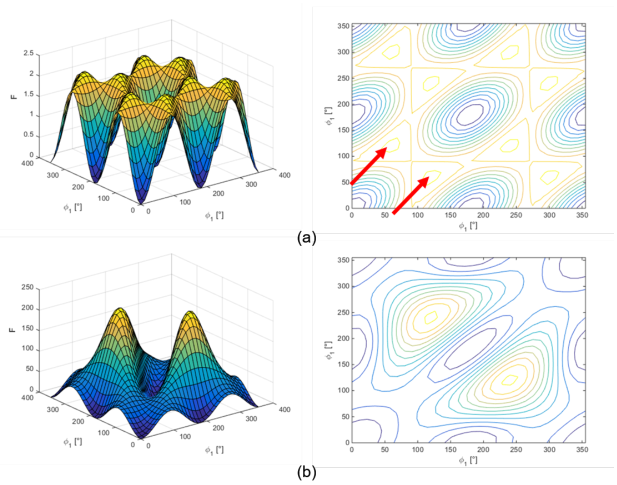


Fig. 7. Surface plot of the objective function in the case of three sensor elements, one sensor element is fixed at $\varphi_0$=0). (a) Uncorrelated random deviations. Clearly, there is no unique optimum. Besides the solution shown in Fig.6, where the separation is $60^o$, also a separation of $120^o$ is optimal. (b) Correlation due to common offset. In this case, only a separation of $120^o$ is optimal. In both cases, the target function is non-convex so that in the general case is not guaranteed that the numeric optimization procedure will find the global optimum.

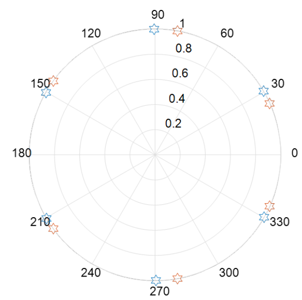


Fig. 8. Example location of three magnetic field sensors (marked as pentagrams). For the given signal model, the uniform distribution of the sensors is actually optimal. However, this does not hold for other numbers of magnetic field sensors.

TABLE I
RESULTS FOR DIFFERENT SENSOR CONFIGURATIONS USING THE I/Q APPROACH AND THE MINIMAX APPROACH. EXCEPT IN CASE OF CORRELATIONS DESCRIBED BY A RANDOM FIELD AND ONLY TWO MAGNETIC FIELD SENSORS, THE UNIFORM/0/90 DEGREE DISTRIBUTION OF SENSOR ELEMENTS IS ACTUALLY OPTIMAL. THE MINIMAX APPROACH DOES NOT PROVIDE USEFUL RESULTS FOR THE CASE WITH ONLY TWO SENSORS. IN THE CASE OF MORE SENSORS IT TURNS OUT THE THE OPTIMUM SOLUTION IS NON-UNIFORM IN THE CASE OF CORRELATIIONS DESCRIBED BY A RANDOM FIELD.

| | | I/Q Criterion | | θ Criterion | |
|---|---|---|---|---|---|
| | Model | Type | 0/90/uniform | Type | 0/90/uniform |
| 2 sensor elements | uncorrelated | analytic | yes | analytic | fail (180°) |
| | common offset | analytic | yes | analytic | fail (180°) |
| | Random Field | numeric | no | numeric | fail (180°) |
| more sensor elements | uncorrelated | numeric | yes/not unique | numeric | yes |
| | common offset | numeric | yes | numeric | yes |
| | Random Field | numeric | yes | numeric | no |

that we obtain a uniform distribution of the sensors when the I/Q approach is used. However, the minimax approach on the angle $\theta$ actually gives different results. With respect to the error of $\theta$ both designs have practically equal performance when we use the estimator according to (15). Although no significant reduction of uncertainty is achieved, this means that non-uniform distributions of sensors can achieve the same performance as uniform distributions. As the restriction to use such regular designs can be omitted, new design options are provided, which may allow to make better use of the area of the silicon die.

The results for the different scenarios are summarized in Table I.

## VI. Conclusion

We present a fast approach to find D-optimal designs for angular position sensors. Optimization results show that commonly used designs of magnetic field sensors are actually optimal for the determination of inphase and quadature contributions in the case of uncorrelated measurements. However, these designs are not the only optimal designs, which opens up the design space. In the presence of spatial correlations, it was found that the uniform sensor distribution is optimal with respect to the determination of inphase and quadrature contributions when the number of sensors is higher or equal to three. It was also found that other designs (e.g. obtained using the minimax approach with respect to the determinateion of the angle) achieve practically equivalent performance also in this case, which again opens up the space of practiacally useful designs.

## APPENDIX

For certain simplified cases, analytic solutions can be found. This holds for two sensor elements, uncorrelated measurements and a parameter vector $\boldsymbol{\eta}$ only comprising shift angles, such that

$$S(\theta, \varphi) = A \cdot sin(\theta + \varphi) + w = h(\theta, \varphi) + w \quad (26)$$

The I/Q model becomes

$$S(\theta, \varphi) = Q_A(\theta) \cdot \cos(\varphi) + I_A(\theta) \cdot \sin(\varphi) + w \quad (27)$$

In this case $I_A$ would represent the cosine and $Q_A$ the sine-part of the measurement angle. For design optimisation we need the derivative of the (spatial) signal $S$ with respect to the parameter of interest and we obtain

$$H = \frac{dh(\theta)}{d\theta} = \begin{bmatrix} sin(\varphi) & cos(\varphi) \end{bmatrix} \quad (28)$$

and the model can be written as

$$\mathbf{S}(\boldsymbol{\theta}, \boldsymbol{\varphi}) = \mathbf{H} \begin{bmatrix} Q_A \\ I_A \end{bmatrix} + w \quad (29)$$

with $\boldsymbol{\theta} = \begin{bmatrix} Q_A & I_A \end{bmatrix}^T$. With two sensor elements we obtain

$$\mathbf{H} = \frac{dh(\theta)}{d\theta} = \begin{bmatrix} sin(\varphi_1) & cos(\varphi_1) \\ sin(\varphi_2) & cos(\varphi_2) \end{bmatrix} \quad (30)$$

In the desired case to find the most optimum angular positions for these two angular sensors we need to find the Fisher-Information $I(\theta)$ which has to be maximum:

$$\mathbf{I}(\theta) = \frac{H^T \cdot H}{\sigma^2} \quad (31)$$

Since we only need the maximum of $\mathbf{I}$, $\sigma$ is only a scaling factor and can be omitted for this optimization and take the determinant:

$$det(H^T \cdot H) = 1 - cos(2\varphi_1 - 2\varphi_2) \quad (32)$$

The maximum is given when $\varphi_1$ and $\varphi_2$ have a difference of $\frac{\pi}{2}$, such that the cosine function has a value equal to -1 and $det(H^T \cdot H) = 1 - cos(2\varphi_1 - 2\varphi_2)$ attains the maximum value of 2.


## ACKNOWLEDGEMENT

This project has received funding from the Electronic Component Systems for European Leadership Joint Undertaking under grant agreement No 692480 (IoSense). This Joint Undertaking receives support from the European Unions Horizon 2020 research and innovation programme and Germany, Netherlands, Spain, Austria, Belgium, Slovakia.


## REFERENCES


[1] A. Arami, J. D. Rechenmann, and K. Aminian, “Reference-free automated magnetic sensor calibration for angle estimation in smart knee prostheses,” *IEEE Sensors Journal*, vol. 14, no. 6, pp. 1788–1796, June 2014.

[2] U. Ausserlechner, “The optimum layout for giant magneto-resistive angle sensors,” *IEEE Sensors Journal*, vol. 10, no. 10, pp. 1571–1582, Oct. 2010.

[3] H. Yamazaki, H. Hirabayashi, N. Oyama, and M. Sakai, “Characterisitics of TMR angle sensors,” *Proceedings SENSOR 2011*, pp. 361–365, 2011.

[4] S. T. Wu, J. Y. Chen, and S. H. Wu, “A rotary encoder with an eccentrically mounted ring magnet,” *IEEE Transactions on Instrumentation and Measurement*, vol. 63, no. 8, pp. 1907–1915, Aug 2014.

[5] V. Fedorov and P. Hackl, *Model-Oriented Design of Experiments*, ser. Lecture Notes in Statistics, P. Bickel, P. DIggle, S. Fienberg, K. Krickeberg, I. Olkin, N. Wermuth, and S. Zeger, Eds. NY, USA: Springer, 1997, vol. 125.

[6] Z. Zhang and L. D. Rienzo, “Optimization of magnetic sensor arrays for current measurement based on swarm intelligence and d-optimality,” *COMPEL - The international journal for computation and mathematics in electrical and electronic engineering*, vol. 28, no. 5, pp. 1179–1190, 2009.

[7] E. G. Ryan, C. C. Drovandi, J. M. McGree, and A. N. Pettitt, “A review of modern computational algorithms for bayesian optimal design,” *International Statistical Review*, vol. 84, no. 1, pp. 128–154, 2016, 10.1111/insr.12107. [Online]. Available: http://dx.doi.org/10.1111/insr.12107

[8] H. Zangl and S. Mühlbacher-Karrer, “Artefact reduction in fast bayesian inversion in electrical tomography,” *COMPEL: The International Journal for Computation and Mathematics in Electrical and Electronic Engineering*, vol. 34, no. 5, pp. 1381–1391, 2015.

[9] B. Amzal, F. Y. Bois, E. Parent, and C. P. Robert, “Bayesian-optimal design via interacting particle systems,” *Journal of the American Statistical Association*, vol. 101, no. 474, pp. 773–785, 2006. [Online]. Available: http://dx.doi.org/10.1198/016214505000001159

[10] C. P. Kitsos, *Optimal Experimental Design for Non-Linear Models*, ser. SpringerBriefs in Statistics. Springer-Verlag Berlin Heidelberg, 2013.

[11] J. Nelder and R. Mead, “A simplex method for function minimization,” *Comput. J*, vol. 7, pp. 308–313, 1965.

[12] Y. Kanda, “Piezoresistance effect of silicon,” *Sensors and Actuators A: Physical*, vol. 28, no. 2, pp. 83 – 91, 1991. [Online]. Available: http://www.sciencedirect.com/science/article/pii/092442479185017I

[13] B. Haelg, “Piezo-hall coefficients of n-type silicon,” *Journal of Applied Physics*, vol. 64, no. 1, pp. 276–282, 1988. [Online]. Available: http://scitation.aip.org/content/aip/journal/jap/64/1/10.1063/1.341422

[14] H. Husstedt, U. Ausserlechner, and M. Kaltenbacher, “In-situ analysis of deformation and mechanical stress of packaged silicon dies with an array of hall plates,” *IEEE Sensors Journal*, vol. 11, no. 11, pp. 2993–3000, Nov 2011.

[15] M. Motz and U. Ausserlechner, *Electrical Compensation of Mechanical Stress Drift in Precision Analog Circuits*. Cham: Springer International Publishing, 2017, pp. 297–326.

[16] C. E. Rasmussen and C. K. I. Williams, *Gaussian Processes for Machine Learning (Adaptive Computation and Machine Learning)*. The MIT Press, December 2006.